IDENTIFYING BROAD AND NARROW FINANCIAL RISK FACTORS VIA CONVEX OPTIMIZATION: PART II

Alexander D. Shkolnik
ads2@berkeley.edu

joint with Jeffrey Bohn and Lisa Goldberg.
Overview

• Simulations (synthetic data).
• Performance metrics (finance).
  – benchmark: PCA.
• Theoretical considerations for $N > T$. 
Simulations (synthetic data)

- $N = 125$ and $N = 500$ securities.
- $T = 250$ observations (one year of daily data).
  - $K = 2$ broad factors ($16\%$ ann. vol., $4\%$ ann. vol.)
  - $\kappa \approx N / \log N$ and $\kappa \approx \sqrt{N}$ countries.
- 400 simulations (realizations of a sample covariance matrix).
- Convex program parameters: $\theta = (\gamma, \lambda)$,
  - $\gamma = 1 / \sqrt{N}$ (see Candès, Li, Ma & Wright (2011)),
  - $\lambda$ set to to control the number of recovered broad factors.
Input to algorithm ($N = 125$, $T = 250$)
Ordered by country
True vs recovered broad factor matrix $L_K$
True vs recovered narrow factor matrix $L_\kappa$
True vs recovered specific risk matrix
Measuring performance on simulated data

- Typically, for estimator $\Sigma(\theta)$, consider $\|\Sigma(\theta) - \Sigma\|$ for some norm.
- Instead, consider the variance of returns to portfolio $w$.

\[
\text{Var}_\Sigma(w) = w^\top \Sigma w
\]  
\[\text{(1)}\]

- The portfolio risk forecasting ratio is computed as

\[
\mathcal{R}_\Sigma(w | \theta) = \frac{\text{Var}_{\Sigma(\theta)}(w)}{\text{Var}_\Sigma(w)}.
\]  
\[\text{(2)}\]

- Ratios $\mathcal{R}_{L_K}$, $\mathcal{R}_{L_\kappa}$ and $\mathcal{R}_\Delta$ are defined analogously.

- Other estimators: $K(\theta)$ and $\kappa(\theta)$. 
Test portfolios

- Let $e = (1, \ldots, 1)^T$. **Equally weighted** portfolio is given by
  \[
  w = e / N
  \]  
  (3)

- **Minimum variance** (long only) portfolio is the solution $w(\theta)$ of
  \[
  \min_w w^T \Sigma (\theta) w
  \]  
  subject to $w^T e = 1$,  
  (4)  
  \[
  w \geq 0.
  \]  
  (5)  

- Minimum variance (long/short): solves (4) subject to $w^T e = 1$,
  \[
  w(\theta) = \frac{\Sigma^{-1}(\theta) e}{e^T \Sigma^{-1}(\theta) e}.
  \]  
  (7)
### Number of factors: equally weighted portfolio

<table>
<thead>
<tr>
<th>$N$</th>
<th>Metric</th>
<th>DSLR: $(\lambda \times 10^3)$</th>
<th>PCA</th>
<th>Ground Truth</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>$K(\theta)$</td>
<td>8.51</td>
<td>2.45</td>
<td>2.00</td>
</tr>
<tr>
<td></td>
<td>(stdev)</td>
<td>(0.83)</td>
<td>(0.44)</td>
<td>(0.00)</td>
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<tr>
<td></td>
<td>$\kappa(\theta)$</td>
<td>20.3</td>
<td>22.4</td>
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</tr>
<tr>
<td></td>
<td>(stdev)</td>
<td>(0.21)</td>
<td>(0.86)</td>
<td>(0.00)</td>
</tr>
</tbody>
</table>

$\star T = 250$ (observations).
## Performance: equally weighted portfolio

<table>
<thead>
<tr>
<th>$N$</th>
<th>Metric</th>
<th>DSLR: ($\lambda \times 10^3$)</th>
<th>PCA</th>
<th>Ground Truth</th>
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</thead>
<tbody>
<tr>
<td>500</td>
<td>$R_\Sigma$</td>
<td>1.00</td>
<td>0.96</td>
<td>1.06</td>
</tr>
<tr>
<td></td>
<td>(stdev)</td>
<td>(0.08)</td>
<td>(0.08)</td>
<td>(0.08)</td>
</tr>
<tr>
<td></td>
<td>$R_{L_K}$</td>
<td>0.97</td>
<td>0.94</td>
<td>1.18</td>
</tr>
<tr>
<td></td>
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<td>(0.09)</td>
<td>(0.09)</td>
<td>(0.10)</td>
</tr>
<tr>
<td></td>
<td>$R_{L_\kappa}$</td>
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<td>0.68</td>
<td>n/a</td>
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<td></td>
<td>(stdev)</td>
<td>(0.09)</td>
<td>(0.05)</td>
<td>(0.00)</td>
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<td></td>
<td>$R_\Delta$</td>
<td>0.89</td>
<td>1.12</td>
<td>1.16</td>
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<td>(stdev)</td>
<td>(0.01)</td>
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<tr>
<td></td>
<td>Ann. Vol.</td>
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<td>0.02</td>
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</table>
## Performance: optimized portfolio

<table>
<thead>
<tr>
<th>$N$</th>
<th>Metric</th>
<th>DSLR: ($\lambda \times 10^3$)</th>
<th>PCA</th>
<th>Ground Truth</th>
<th>Ann. Vol.</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>$\mathcal{R}_{\Sigma}$</td>
<td>0.97</td>
<td>0.88</td>
<td>0.81</td>
<td>0.11</td>
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<td>(0.06)</td>
<td>(0.07)</td>
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<tr>
<td></td>
<td>$\mathcal{R}_{L_K}$</td>
<td>0.86</td>
<td>0.70</td>
<td>0.83</td>
<td>0.07</td>
</tr>
<tr>
<td></td>
<td>(stdev)</td>
<td>(0.08)</td>
<td>(0.10)</td>
<td>(0.13)</td>
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</tr>
<tr>
<td></td>
<td>$\mathcal{R}<em>{L</em>\kappa}$</td>
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<td>0.48</td>
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</tr>
<tr>
<td></td>
<td>(stdev)</td>
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<td>(0.07)</td>
<td>(0.00)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\mathcal{R}_{\Delta}$</td>
<td>0.96</td>
<td>1.22</td>
<td>1.31</td>
<td>0.08</td>
</tr>
<tr>
<td></td>
<td>(stdev)</td>
<td>(0.02)</td>
<td>(0.03)</td>
<td>(0.05)</td>
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</tbody>
</table>
Summary of findings

- **SLRD** outperforms classical **PCA** on risk forecasts.
- Accurate sparse component recovery implies:
  - accurate $\mathcal{R}_\Delta$ forecast,
  - accurate $\kappa(\theta)$ estimates.
- Inaccurate $K(\theta)$ estimates **does not** imply poor risk forecast.
- $N > T$?
$\kappa (\theta)$ estimates ($N > T$)

- With high probability CPW$^a$ recovers correctly the
  - rank of $\mathcal{L}$ and sparsity support of $S$.
  - Rate of convergence $\sqrt{N/T}$ arises from bounds on
    \[
    \| \mathcal{L} - \mathcal{L} (\theta) \|_2 \text{ and } \| S - S (\theta) \|_{\ell_\infty}. \tag{8}
    \]

- The error on $\| S - S (\theta) \|_{\ell_\infty}$ may be improved to order
  \[
  \sqrt{\log N \over T}. \tag{9}
  \]

- When sparsity pattern is correct, $\kappa (\theta)$ is correct.

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$^a$(Chandrasekaran, Parrilo & Willsky 2012)
K (θ) estimates (N > T)

- With high probability CPW\textsuperscript{a} recovers correctly the
  - rank of \( \mathcal{L} \) and sparsity support of \( S \).
  - Rate of convergence \( \sqrt{N/T} \) arises from bounds on
    \[
    \| \mathcal{L} - \mathcal{L} (\theta) \|_2
    \] (10)

- **Cannot** estimate number of broad factors accurately for \( N > T \).

- Numerical experiments suggest \( \| \mathcal{L} - \mathcal{L} (\theta) \|_{\ell_\infty} \) may be order
  \[
  \sqrt{\log N \over T}.
  \] (11)

\textsuperscript{a}(Chandrasekaran et al. 2012)
Risk forecasts

- **Equally weighted**: accuracy of portfolio risk forecast depends on

\[
\left| \mathcal{R}_\Sigma (w \mid \theta) - 1 \right| \lesssim \| \mathcal{L}(\theta) - \mathcal{L} \|_{\ell_\infty} \wedge \| S(\theta) - S \|_{\ell_\infty} \quad (12)
\]

\[
= O\left( \sqrt{\frac{\log N}{T}} \right). \quad (13)
\]

- **Optimized**: same bound holds with different constants.

- Similar for factor and specific risk forecasts.

- Support for accuracy of risk forecasts in $\log (N) / T \downarrow 0$ regime.
Conclusions

• The approach shows promise in extracting narrow factors that highlight country/industry relationships in data.

• Financial applications require a reformulation of available low rank and sparse decomposition methods.
  – low-rank + sparse + diagonal decomposition,
  – sparse eigenvectors (narrow factors).

• Finance oriented performance metrics may lead to an alternative analysis of estimator consistency.
Ongoing & future work

• Theoretical performance guarantees for risk forecasting ratios.

• Data-driven methods of selecting optimal parameters $\theta = (\lambda, \gamma)$.

• Alternative convex programs guided by performance metrics.

• Scale algorithms to tens of thousands of securities.
Questions.

(ads2@berkeley.edu)
\( R_\Delta \) (specific risk) forecast

- **MTFA**\(^a\) decomposes a given matrix \( S \) into a low rank \( L_\kappa = XGX^T \) component and a diagonal component \( \Delta \) exactly if

\[
\max_{x \in \mathcal{X} / \{0_N\}} \frac{\|x\|_\infty}{\|x\|_2} < 1/2. \tag{14}
\]

where \( \mathcal{X} \) is the space spanned by the narrow factor exposures in \( X \).

- A sufficient condition for (14) is

\[
x_i^2 < \sum_{i \neq j} x_j^2 \quad \text{for all } i \text{ and } x \in \mathcal{X}. \tag{15}
\]

- Robust recovery for factors that are not too narrow.

\(^a\)(Saunderson, Chandrasekaran, Parrilo & Willsky 2012)
References

