PCA with Model Misspecification

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Principal Components Analysis (PCA) in financial data

Assume Return Generating Process of form

\[ R = \phi X + \varepsilon \]  \hspace{1cm} (1)

Security Returns = Factor Returns \times Factor Sensitivities
+ Idiosyncratic Returns, where

\[ N = \text{number of securities} \]
\[ K = \text{number of factors} \]
\[ T = \text{number of return periods (days, ...)} \]

in estimation window

\[ \text{Cov} (\phi) = I_K, \text{ the } K \times K \text{ identity matrix} \]
PCA in financial data

- Compute eigenvalues and eigenvectors of “covariance” matrices
  - $C_{N \times N} = \frac{R^\top R}{T}$ or
  - $C_{T \times T} = \frac{R R^\top}{T}$

or some weighted version (correlation rather than covariance, market cap, inverse volatility, temporal weighting, . . .).

- Typically, use $C_{T \times T}$; we are interested in the eigenvectors of $C_{N \times N}$, but the two matrices have the same nonzero eigenvalues and closely related eigenvectors.

- Throughout, “covariances” and “correlations” are computed without demeaning
  - Follows practitioner literature
  - Expected daily equity returns are very close to zero; the sample mean return over a (one-year) estimation window is a noisy estimate of zero
There is More Information in $R$ than in the “Covariance” matrix $C_{N \times N}$

- Eigenvalues and eigenvectors of $C_{N \times N}$ depend on $R$ only through the “covariance” matrix $C_{N \times N}$
- $x$ is an eigenvector of $C_{N \times N}$
  - (portfolio representation of an estimated factor)
  $\iff Rx$ is an eigenvector of $C_{T \times T}$
  - (return of an estimated factor)
- Eigenvectors of $C_{T \times T}$ contain information about the distribution of factor returns that is not found in the “covariance” matrix $C_{N \times N}$:
  - Gaussian?
  - Excess kurtosis, as in Student $t$ or other power laws?
  - Negative skew?
Chamberlain and Rothschild (1983), Connor and Korajczyk (1988), Bai (2003), …

- Asymptotic theory in which $T, N \to \infty$ so that $\varepsilon$ is not important for diversified portfolios

- Assumes that $C_{N \times N} \sim \frac{X^T \phi^T \phi X}{T}$ converges in an appropriate sense
Variable Volatility in $\phi$: In financial data, volatility changes frequently

- Changes in factor volatility (volatility of $\phi$) *change the correlations of securities*
- Example: An increase in market volatility causes the average correlation between securities to rise

Regimes in $X$: In financial data, the sign of the correlation between assets reverses from time to time

- Example: The correlation between the equity market and the price of oil is generally positive in response to demand shocks and negative in response to supply shocks

Thus, the assumption that $C_{N \times N}$ converges is problematic when applied to financial data
Two Approaches in the Literature

- Pelger (2015a, 2015b), Ait-Sahalia and Xiu (2015): Use intraday data to make $T$ large with a fixed time horizon
  - The horizon is such that there is plausibly only one $X$ regime
  - Replace the covariance matrix with the quadratic covariation process (a matrix-valued stochastic process whose realization at any time is a covariance matrix).
  - Limitation: using intraday data in a global model is problematic due to temporal asynchronicity

- Identify regimes with a Markov switching model
  - Attractive option for $X$ regimes.
  - Unattractive for regimes that only involve variable factor volatility (too many regimes, volatility changes all the time, ...)
    - Not necessary
This Project

- Proposes an alternative for dealing with variable $\phi$ volatility, which is conceptually related to use of the quadratic covariation in Pelger
- Identifies an approach to dealing with non-Gaussian return distributions
- Identifies strengths and weaknesses in PCA applied to data containing $X$ regimes
**Variable Volatility: Formulation**

- **$K$ Constant Volatility Factors** $\tilde{\phi}$, covariance matrix the $K \times K$ identity, IID across time.
- Assume factor distribution is parametrized by a single scale factor. *Need not be Gaussian.*
- Volatility process $v$ taking values in $\mathbb{R}^K$
  - Independent of $\tilde{\phi}$
  - Perhaps generated by a mean-reverting process such as the Heston Model (volatility given by modification of Ornstein-Uhlenbeck process)
- **$K$ Variable Volatility Factors** $\phi$ whose returns on dates $t = 1, \ldots, T$ are given by the Hadamard (elementwise) product of $v$ and $\tilde{\phi}$

\[
\phi = v \circ \tilde{\phi} = \begin{pmatrix}
v_{11}\tilde{\phi}_{11} & \cdots & v_{1K}\tilde{\phi}_{1K} \\
\vdots & \ddots & \vdots \\
v_{T1}\tilde{\phi}_{T1} & \cdots & v_{TK}\tilde{\phi}_{TK}
\end{pmatrix}
\]

- Analogous formulation for idiosyncratic volatility $\varepsilon = v \circ \tilde{\varepsilon}$
Variable Volatility: Conceptual Idea

- $C_{N \times N} \sim X^\top Q X$ where
  - $Q = \frac{\phi^\top \phi}{T}$ is the realized covariance matrix of the factor returns (discrete analogue of quadratic covariation in Pelger)

$$Q \sim D = \begin{pmatrix}
\sigma_1^2 & 0 & \cdots & 0 \\
0 & \sigma_2^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_K^2
\end{pmatrix} \quad (2)$$

where $\sigma_k^2 = \frac{1}{T} \sum_{t=1}^{T} v_{tk}^2$ is the variance (conditional on $v$) of factor $k$ over the period $\{1, \ldots, T\}$.

- Note that $D$ need not converge in any sense.

- The rows of $X$ are eigenvectors of $X^\top D X$, hence are approximate eigenvectors of $C_{N \times N}$, so PCA correctly estimates factor sensitivities even with variable factor volatility

- This is true even though changing volatility changes the correlation of assets
Use PCA over the whole estimation period \( \{1, \ldots, T\} \), without separating different volatility regimes.

This yields two sets of eigenvectors:

- The eigenvectors of \( C_{N \times N} \) are estimates of the factor sensitivities (rows of \( X \)).
- The eigenvectors of \( C_{K \times K} \) are estimates of the time series of the factor profit/loss.

Combine the estimated factor sensitivities (eigenvectors of \( C_{N \times N} \)) with exponentially weighted (short half-life) standard deviation of the eigenvectors of \( C_{K \times K} \). **Responsive Volatility Adjustment**
Errors in predictions of portfolio volatility are modestly higher than in the constant volatility case.

The errors in the estimated rows of $X$ are higher than in the constant volatility case. Further study of the economic significance is needed.

Gaussian estimates of Value at Risk (VaR) (e.g. lower 3% quantile of return) substantially underpredict risk in the presence of negative skewness.

Gaussian estimates of Expected Tail Loss (ETL) (e.g. conditional expectation of loss over lower 3% quantile of return) substantially underpredict risk in the presence of negative skewness and/or excess kurtosis.
Historical Method for Predicting VaR and ETL

- Compute past distribution of Z-scores of portfolio return (actual return divided by predicted return volatility)
- Use these to predict tomorrow’s VaR and ETL, conditional on today’s volatility prediction
- In simulation,
  - Out-of-sample estimates of VaR using Historical Method are much more accurate than Gaussian estimates in simulation
  - Out-of-sample estimates of ETL using Historical Method are much more accurate than Gaussian estimates in the absence of skewness
  - With negative skewness, Historical Method overpredicts ETL, while Gaussian methods underpredict ETL. Looking for ways to correct overprediction.
### Table 1: Variable Volatility vs. Constant Volatility

<table>
<thead>
<tr>
<th>Half-Life</th>
<th>Variable Volatility</th>
<th>Constant Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bias</td>
<td>DD</td>
</tr>
<tr>
<td></td>
<td>Factor 1</td>
<td>Factor 2</td>
</tr>
<tr>
<td>10</td>
<td>1.029</td>
<td>0.029</td>
</tr>
<tr>
<td>20</td>
<td>1.013</td>
<td>0.029</td>
</tr>
<tr>
<td>30</td>
<td>1.008</td>
<td>0.029</td>
</tr>
<tr>
<td>40</td>
<td>1.007</td>
<td>0.029</td>
</tr>
<tr>
<td>50</td>
<td>1.008</td>
<td>0.029</td>
</tr>
<tr>
<td>∞</td>
<td>1.042</td>
<td>0.029</td>
</tr>
</tbody>
</table>

**Table:** Performance of Standard PCA with Responsive Volatility Adjustment in Variable and Constant Factor Volatility Models. The underlying constant-volatility model is the Bianchi, Goldberg and Rosenberg (2016) two-factor model. This table reports Bias, and average Directional Distance between the estimated and true factors, with \( N = 1,000 \) stocks, a PCA estimation window of \( T = 250 \) days, and 50,000 Iterations. Bias is calculated for the Equally-Weighted Portfolio; Directional Distance gives guidance for smaller or optimized portfolios.
Table 2: Gaussian and Historical Method VaR Exceedances and ETL

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Volatility</th>
<th>Gaussian 3% VaR Exceed</th>
<th>Gaussian Predictions Ratio</th>
<th>Historical 3% VaR Exceed</th>
<th>Historical Method 3% ETL Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>Constant</td>
<td>3.000%</td>
<td>-1.011</td>
<td>2.956%</td>
<td>-1.001</td>
</tr>
<tr>
<td>Gaussian</td>
<td>Variable</td>
<td>3.246%</td>
<td>-1.071</td>
<td>2.987%</td>
<td>-1.001</td>
</tr>
<tr>
<td>Student t</td>
<td>Variable</td>
<td>3.292%</td>
<td>-1.186</td>
<td>2.971%</td>
<td>-1.001</td>
</tr>
<tr>
<td>Skew</td>
<td>Variable</td>
<td>4.376%</td>
<td>-1.252</td>
<td>2.956%</td>
<td>-0.725</td>
</tr>
</tbody>
</table>

Table: Simulated Predicted 3% VaR Exceedance and Ratio of 3% ETL to Predicted 3% ETL. Predictions are derived from estimated volatility, using either Gaussian assumptions or the Historical Method. Simulation with \( N = 1,000 \) stocks, \( T = 250 \) days in each PCA window, and 50,000 Iterations, using the Bianchi, Goldberg and Rosenberg (2016) two-factor model. The underlying constant-volatility factor returns are either Gaussian with constant volatility; or Gaussian, Student t, or skewed with variable volatility. This is a two-factor model in which both factors follow the same discrete version of the Heston Process. Volatility of the Equally-Weighted Portfolio is predicted with an exponential 40-day half-life. Var and ETL predictions are then made using either Gaussian assumptions or the Historical Method.
X Regimes

Theorem

If we apply PCA to a data history combining two different X regimes, then

- A factor which is present in both regimes will be identified as an eigenvector.
- A factor which is present in Regime I and not in Regime II will be identified as an eigenvector if and only if it is orthogonal to all the factors present in Regime II.

In particular, a factor which is present only one of the regimes is likely to be “hybridized” with a factor in the other regime, rather than being cleanly identified by PCA.

Ait-Sahalia, Yacine and Dacheng Xiu, “Principal Components Analysis of High-Speed Data,” technical report, University of Chicago.


