# Bootstrapping $r$-Fold Tensor Data

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# The IID bootstrap

- data are IID $F$
- we resample IID from the empirical distribution $\widehat{F}$
- getting variance estimates and confidence intervals

## We like it because

- face value validity (or at least explainability)
- deep theory for $\bar{X}$ vs. $\mathbb{E}(X)$
- extensions to more general statistics

Bootstrap (and cross-validation) let us use very mild assumptions:

1) IID data, and

2) non-pathological moments.

# IID data vectors

| | Variable 1 | $\cdots$ | Variable C |
|---|---|---|---|
| Case 1 | | | |
| $\vdots$ | | | |
| Case R | | | |

1) Variables are named entities:
   - E.g. pressure, volume, income $\cdots$
   - They persist
2) Cases are anonymous replicates
   - Sampled IID from some $F$
   - Of no inherent interest
   - We'd rather just know $F$

## For IID data $\cdots$

. . . we only care about cases because they show relationships among variables.

# Two-way data

| Rating | Viewer 1 | Viewer 2 | Viewer 3 | $\cdots$ | Viewer C |
|---|---|---|---|---|---|
| Movie 1 | 4 | 4 | 1 | $\cdots$ | 4 |
| Movie 2 | 5 | 5 | NA | $\cdots$ | NA |
| Movie 3 | 3 | 3 | NA | $\cdots$ | 2 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| Movie R | NA | 5 | 3 | $\cdots$ | 4 |

More examples of two-way data:

| | | | | |
|---|---|---|---|---|
| genes | $\times$ | environments | $\longrightarrow$ | crop yields |
| terms | $\times$ | documents | $\longrightarrow$ | counts |
| candidate | $\times$ | interviewer | $\longrightarrow$ | rating |
| nodes | $\times$ | more nodes | $\longrightarrow$ | labeled edges |

# Tensor data

$r$-way data, i.e. an $r$-tuple of named entities. For example:

| | |
|---|---|
| Suppose that | customer $U$ |
| comes from | computer (machine) $M$ |
| enters | query $Q$ |
| reads | review $R$ |
| buys | book $B$ |
| with credit card | book $C$ |
| ships to | address $A$ |

Then Amazon's logs get $(U, M, Q, R, B, C, A)$ among other variables (such as price paid).

While $r = 2$ is most common, $r > 2$ arises frequently.

# Tuples

| | Movie | Viewer | Rating |
|---|---|---|---|
| Case 1 | 1 | 1 | 4 |
| Case 2 | 1 | 2 | 4 |
| Case 3 | 2 | 1 | 5 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| Case N | R | C | 4 |

- Now cases are anonymous
- We don't store the NAs
- $2$ categorical variables
  with lots of levels
- Not independent:
  - Cases 1 & 2 share a movie
  - Cases 1 & 3 share a viewer

How should we bootstrap and cross-validate data like this?

What about $r > 2$?

Maybe large $N$ means no meaningful uncertainty.

# Random effects model

$$X_{ij} = \mu + a_i + b_j + \varepsilon_{ij} \qquad i = 1, \dots, R \qquad j = 1, \dots, C$$

$a_i \sim \mathcal{N}(0, \sigma_A^2)$ e.g. plants

$b_j \sim \mathcal{N}(0, \sigma_B^2)$ e.g. environments

$\varepsilon_{ij} \sim \mathcal{N}(0, \sigma_E^2)$

Used in agriculture

Studied for decades

$\hat{\mu}$ is $\bar{X}_{\bullet\bullet}$

No bootstrap exists for $V(\hat{\mu})$

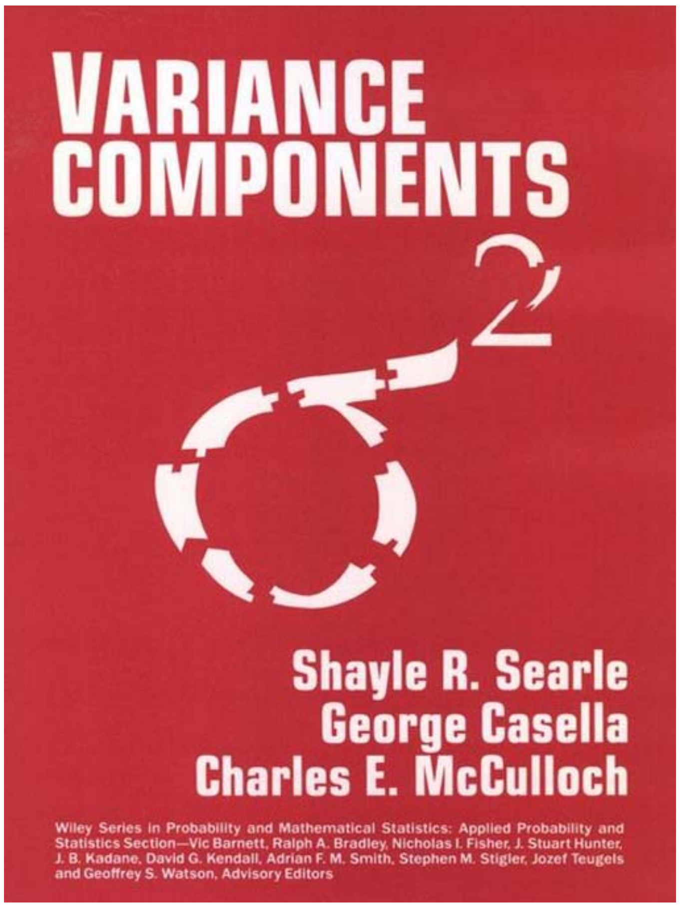
None can exist $\cdots$

$\cdots$ McCullagh (2000)

We can't even bootstrap a balanced $\bar{X}$ !

# What about classical approaches?

prime reference:




- Excellent for balanced Gaussian data
- Unbalance $\Longrightarrow$ invert large matrices
- Emphasis on homogeneous variances

# McCullagh (2000)

For $\hat{\mu} = \bar{X}_{\bullet\bullet} = \frac{1}{R}\frac{1}{C}\sum_{i=1}^{R}\sum_{j=1}^{C} X_{ij}$

**Boot-I** Resample from $N = RC$ values

**Boot-II** Resample $R$ rows and resample $C$ columns (indep)

$$V(\hat{\mu}) = \frac{\sigma_A^2}{R} + \frac{\sigma_B^2}{C} + \frac{\sigma_E^2}{RC} \qquad \text{true var}$$

$$\mathbb{E}(\hat{V}_{\mathrm{I}}(\hat{\mu})) \doteq \left(\sigma_A^2 + \sigma_B^2 + \sigma_E^2\right)\frac{1}{RC} \qquad \text{way too small}$$

$$\mathbb{E}(\widehat{V}_{\mathrm{II}}(\hat{\mu})) \doteq \frac{\sigma_A^2}{R} + \frac{\sigma_B^2}{C} + \frac{3\sigma_E^2}{RC} \qquad \text{not so bad}$$

Boot-I is seriously flawed, Boot-II is close

# The case $r = 2$

O (2007)

Independent bootstrap of rows and columns

Allows for missing data $\cdots$ but conditions on pattern of observed data

Allows non-homogeneous $V(a_i)$, $V(b_j)$ and $V(\varepsilon_{ij})$

Still get $\mathbb{E}(\widehat{V}_{\mathrm{B}}(\hat{\mu})) \doteq V(\hat{\mu})$, i.e.

Still get $\approx 1 \times$ the main effect contribution
$\approx 3 \times$ the interaction contribution

On Netflix data ... naive bootstrap can under-estimate variance by $56{,}200$ fold

Sunday vs. Tuesday edge of $0.02$ stars is real

mimics pigeonhole model of Cornfield & Tukey (1956)

Fine print:

uniform bounds on variances, and

no row/column has more than $\epsilon$ of the data

# Goals

We would like to get an approximate bootstrap for arbitrary data patterns with $r \geqslant 2$.

We focus on getting the variance approximately right.

| Challenge | Today |
|---|---|
| What happens to that $3$ for $r > 2$? | ● |
| There are many missing data values. | ● |
| Missingness might be informative. | ● |
| The entities might have unequal variances. | ● |
| We might want a little more than $\bar{X}$. | ● |
| We might want a lot more than $\bar{X}$. | ● |

# Illustrative data sets

## Netflix

$N = 100{,}480{,}507$ ratings,

by $480{,}189$ customers,

on $17{,}770$ movies

$X$ is $1$ to $5$ stars

used in famous contest

## Facebook

$18{,}134{,}419$ comments

by $8{,}078{,}531$ commenters

on $2{,}085{,}639$ URLs

shared by $3{,}904{,}715$ sharers

$X$ is $\log($# chars in comment$)$

## Example

**Alice** (shares a URL) “Hey, check out `http://stat.stanford.edu`”

**Bob** (comments on it) “Thanks for sharing that, I learned a lot.”

**Data** `url` $=$ `http://stat.stanford.edu`

`sharer` $=$ Alice

`commenter` $=$ Bob

`log length` $X = \log(41) \doteq 3.71$

# Random effects: $r$-way case

| | |
|---|---|
| Index | $\boldsymbol{i} = (i_1, i_2, \dots, i_r) \in \{1, 2, 3, \cdots\}^r$ |
| Sub-index | $\boldsymbol{i}_u = (i_{j_1}, \dots, i_{j_L}) \quad u = \{j_1, \dots, j_L\} \subseteq \{1, 2, \dots, r\}$ |
| Data | $X_{\boldsymbol{i}} \in \mathbb{R}^d$ short for $X_{i_1, i_2, \dots, i_r}$ use $d = 1$ |
| Presence | $Z_{\boldsymbol{i}} \in \{0, 1\}$ |

We model a random effect for each non-empty $u \subseteq \{1, 2, \dots, r\}$.

$$X_{\boldsymbol{i}} = \mu + \sum_{u \neq \varnothing} \varepsilon_{\boldsymbol{i},u}$$

$$\mathbb{E}(\varepsilon_{\boldsymbol{i},u}) = 0$$

$$\mathrm{Cov}(\varepsilon_{\boldsymbol{i},u}, \varepsilon_{\boldsymbol{i}',u'}) = \sigma^2_{\boldsymbol{i},u} 1_{u=u'} 1_{\boldsymbol{i}_u = \boldsymbol{i}'_u}$$

Homogeneous special case

$$\sigma^2_{\boldsymbol{i},u} \equiv \sigma^2_u \quad \forall\, \boldsymbol{i} \in \mathbb{N}^r \quad \forall\, u \subseteq \{1, \dots, r\}$$

# The product reweighted bootstrap

$$\hat{\mu} = \frac{\sum_{\boldsymbol{i}} Z_{\boldsymbol{i}} X_{\boldsymbol{i}}}{\sum_{\boldsymbol{i}} Z_{\boldsymbol{i}}} \quad \text{and} \quad \hat{\mu}^* = \frac{\sum_{\boldsymbol{i}} Z_{\boldsymbol{i}} W_{\boldsymbol{i}} X_{\boldsymbol{i}}}{\sum_{\boldsymbol{i}} Z_{\boldsymbol{i}} W_{\boldsymbol{i}}}$$

## Our reweighting

$$W_{\boldsymbol{i}} = \prod_{j=1}^{r} W_{j,i_j}$$

$$\mathbb{E}(W_{j,i_j}) = 1 \quad \text{all indep.}$$

$$V(W_{j,i_j}) = \tau^2 \quad \text{usually } \tau^2 = 1$$

# Resampling vs. reweighing

| Bootstrap | Distribution of $W_{j,i_j}$ | Reference |
|---|---|---|
| Original | Multinomial$(N_j; 1/N_j, \ldots, 1/N_j)$ | Efron (1979) |
| Bayesian | $W_{j,i_j} \overset{\text{iid}}{\sim}$ Exp$(1)$ | Rubin (1981) |
| Poisson | $W_{j,i_j} \overset{\text{iid}}{\sim}$ Poi$(1)$ | Oza (2001)<br>Lee & Clyde (2004) |
| Half sampling | $W_{j,i_j} \overset{\text{iid}}{\sim} \mathbf{U}\{0, 2\}$ | McCarthy (1969) |

Independent weights are much simpler to analyze and implement:

data may be spread over servers, countries, continents.

# Joys of half-sampling

$$W_{\boldsymbol{i}} = \prod_{j=1}^{d} W_{j,i_j} \quad \text{where} \quad W_{j,i_j} \stackrel{\text{iid}}{\sim} \mathbf{U}\{0,2\}$$

Original context was stratified sampling, $n = 2$ per stratum.

## As a bootstrap

- All data get integer weights
- All nonzero weights are equal
- Has minimal kurtosis subject to mean $=$ variance $=$ 1.

Each bootstrap computation is the same as the original one but with about $2^{-r}N$ observations.

# True variance (homog. case)

Recall

$$X_{\boldsymbol{i}} = \mu + \sum_{u \neq \varnothing} \varepsilon_{\boldsymbol{i},u}$$

$$V(\varepsilon_{\boldsymbol{i},u}) = \sigma_u^2, \quad \text{and let}$$

$$N \equiv \sum_{\boldsymbol{i}} Z_{\boldsymbol{i}}.$$

Then

$$\hat{\mu} = \frac{1}{N} \sum_{\boldsymbol{i}} Z_{\boldsymbol{i}} X_{\boldsymbol{i}}$$

$$V_{\text{RE}}(\hat{\mu}) = \frac{1}{N^2} \sum_{u \neq \varnothing} \Bigl( \sum_{\boldsymbol{i}} \sum_{\boldsymbol{i}'} Z_{\boldsymbol{i}} Z_{\boldsymbol{i}'} 1_{\boldsymbol{i}_u = \boldsymbol{i}'_u} \Bigr) \sigma_u^2 \equiv \frac{1}{N} \sum_{u \neq \varnothing} \nu_u \sigma_u^2$$

$$\nu_u \equiv \frac{1}{N} \sum_{\boldsymbol{i}} \sum_{\boldsymbol{i}'} Z_{\boldsymbol{i}} Z_{\boldsymbol{i}'} 1_{\boldsymbol{i}_u = \boldsymbol{i}'_u}$$

# Our examples

$$V_{\text{RE}}(\hat{\mu}) \equiv \frac{1}{N}\sum_{u\neq\varnothing} \nu_u \sigma_u^2$$

$$\doteq \frac{1}{N}\Big(56{,}200\sigma^2_{\text{movies}} + 646\sigma^2_{\text{viewers}} + \sigma^2_{\text{interaction}}\Big) \quad \text{(for Netflix)}$$

For Facebook

$$\nu_{\text{sh}} \doteq 17.71, \quad \nu_{\text{com}} \doteq 7.71, \quad \nu_{\text{url}} \doteq 26{,}854.92 \quad !$$

$$\nu_{\text{sh,com}} \doteq 5.92, \quad \nu_{\text{sh,url}} \doteq 12.91, \quad \nu_{\text{com,url}} \doteq 5.19, \quad \text{and}$$

$$\nu_{\text{sh,com,url}} \doteq 4.88.$$

$$\nu_{\text{url}} \geqslant 26{,}000$$

# Naive bootstrap (homog. case)

$$V_{\text{RE}}(\hat{\mu}) = \frac{1}{N} \sum_{u \neq \varnothing} \nu_u \sigma_u^2$$

$$\mathbb{E}_{\text{RE}}(V_{\text{NB}}(\hat{\mu}^*)) = \frac{1}{N} \sum_{u \neq \varnothing} \Bigl(1 - \frac{\nu_u}{N}\Bigr) \sigma_u^2$$

O and Eckles (2011)

Typically $1 \ll \nu_u \ll N$ for $u \neq \{1, \dots, r\}$

Note: $V_{\text{NB}}(\hat{\mu}^*)$ is what the bootstrap settles down to in $B \to \infty$ resamplings.

# Product bootstrap

$$\hat{\mu}^* = \frac{\sum_i Z_i W_i X_i}{\sum_i Z_i W_i} \equiv \frac{T^*}{N^*} \qquad \text{(ratio estimator)}$$

$$V_{\text{PW}}(\hat{\mu}^*) \approx \widetilde{V}_{\text{PW}}(\hat{\mu}^*) \equiv \frac{1}{N^2}\mathbb{E}_{\text{PW}}\bigl((T^* - \hat{\mu}N^*)^2\bigr) \qquad \text{(as } B \to \infty\text{)}$$

The delta method is reliable for large data

(Chamandy, Muralidharan, Najmi (2011))

## Main result

$$\mathbb{E}_{\text{RE}}(\widetilde{V}_{\text{PW}}(\hat{\mu}^*)) = \frac{1}{N}\sum_{u \neq \varnothing} \gamma_u \sigma_u^2$$

where $\gamma_u \approx \nu_u$ if $|u| = 1,$ (i.e. cardinality $1$)

otherwise small $\gamma_u/\nu_u > 1$

# Exact formula depends on

| Notation | Definition | Meaning |
| --- | --- | --- |
| $N_{\boldsymbol{i},u}$ | $\sum_{\boldsymbol{i}'} Z_{\boldsymbol{i}'} 1_{\boldsymbol{i}_u = \boldsymbol{i}'_u}$ | Match $\boldsymbol{i}$ in $u$ |
| $\nu_u$ | $N^{-1} \sum_{\boldsymbol{i}} Z_{\boldsymbol{i}} N_{\boldsymbol{i},u}$ | Avg # matches on $u$ |

Exact result $\gamma_u = \sum_{k=0}^{r} (1+\tau^2)^k (\nu_{k,u} - 2\widetilde{\nu}_{k,u} + \rho_k \nu_u)$ non-asymptotic

$$\mathbb{E}_{\text{RE}}(\widetilde{V}_{\text{PW}}(\hat{\mu}^*)) = \frac{1}{N} \sum_{u \neq \varnothing} \gamma_u \sigma_u^2$$

Fine print from article

- $\nu_{k,u}$ depends on the number of $\boldsymbol{i}$, $\boldsymbol{i}'$ pairs that match in precisely $k$ indices, including those in $u$.
- $\widetilde{\nu}_{k,j}$ depends on the number of triples $\boldsymbol{i}$, $\boldsymbol{i}'$, $\boldsymbol{i}''$ where $\boldsymbol{i}$ matches $\boldsymbol{i}'$ in the set $u$ and matches $\boldsymbol{i}''$ in precisely $k$ indices.

# Approximations

The exact formula captures some bad cases. We can often simplify them.

## Extreme level duplication

e.g. $N/2$ obs in row $1$ and $N/2$ obs in col $1$.

effective sample size is about one or two

Formulas simplify if level duplication is not extreme.

## Variable duplication

Almost every record that matches on some variables matches on a superset of those variables

e.g. match name and phone number $\Rightarrow$ usually match fax number

match age and zip code $\not\Rightarrow$ match occupation

# Duplication indices

(level dup) $$\epsilon = \max_{\boldsymbol{i}} \max_{u \neq \varnothing} \frac{N_{\boldsymbol{i},u}}{N} = \max_{\boldsymbol{i}} \max_{1 \leqslant j \leqslant r} \frac{N_{\boldsymbol{i},\{j\}}}{N}$$

(variable dup) $$\eta = \max_{\varnothing \subsetneq u \subsetneq v} \frac{\nu_v}{\nu_u} = \max_{\varnothing \subsetneq u \subsetneq v} \frac{\nu_v}{\nu_u}$$

## Examples

| | $\epsilon$ | $\eta$ |
|---|---|---|
| Netflix | $\frac{232{,}944}{100{,}480{,}507} \doteq 0.00232$ | $\frac{1}{646} \doteq 0.00155$ |
| | Miss Congeniality | $\nu_{\text{interaction}} / \nu_{\text{movies}}$ |
| Facebook | $\frac{686{,}990}{18{,}134{,}419} \doteq 0.0379$ | $\frac{4.88}{5.19} \doteq 0.94$ |
| | a popular URL | $\nu_{\text{sh,com,url}} / \nu_{\text{com,url}}$ |

$\eta$ is not small for the Facebook data

bootstrap variances will be somewhat more conservative

# Approximations

**Theorem 1.** *In the homogeneous random effects model, the product weight bootstrap with* $V(W_{j,i_j}) = \tau^2 = 1$, *satisfies*

$$\gamma_u = \nu_u[2^{|u|} - 1 + \Theta_u \epsilon] + \sum_{v \supsetneq u} 2^{|v|} \nu_v,$$

*where* $|\Theta_u| \leqslant 2^{r+1} - 2$.

*Proof.* O & Eckles (2011), who consider general $\tau^2$. $\square$

For small $\epsilon$ and $r$ (i.e. $2^r \epsilon \ll 1$)

$$\gamma_u \approx (2^{|u|} - 1)\nu_u + \sum_{v \supsetneq u} 2^{|v|} \nu_v$$

If also $\eta \ll 1$

$$\gamma_u \approx (2^{|u|} - 1)\nu_u$$

# Some specific approximations

For $r = 2$

$$\begin{aligned} \gamma_{\{j\}} &= \nu_{\{j\}}(1 + \Theta_j \epsilon) + 2 \quad j = 1, 2 \\ \gamma_{\{1,2\}} &= \nu_{\{1,2\}}(3 + \Theta_{\{1,2\}}\epsilon), \quad \text{where} \\ |\Theta_u| &\leqslant 6. \end{aligned}$$

For $r = 3$

$$\begin{aligned} \gamma_{\{1\}} &\approx \nu_{\{1\}} + 4\nu_{\{1,2\}} + 4\nu_{\{1,3\}} + 8 \\ \gamma_{\{1,2\}} &\approx 3\nu_{\{1,2\}} + 8 \\ \gamma_{\{1,2,3\}} &\approx 7. \end{aligned}$$

If $0 < m \leqslant \min_u \sigma_u^2 \leqslant \max_u \sigma_u^2 \leqslant M < \infty$ then

$$\frac{\mathbb{E}_{\text{RE}}(\widetilde{V}_{\text{PW}}(\hat{\mu}^*))}{V_{\text{RE}}(\hat{\mu})} = 1 + O(\eta + \epsilon).$$

# Facebook loquacity

For each commenter, url and sharer, we obtain:

$X = \log($#char in comment$)$ as well as,

country $c \in \{$US, UK$\}$ of commenter, and

mode $m \in \{$web, mobile$\}$ of commenter.

Now let

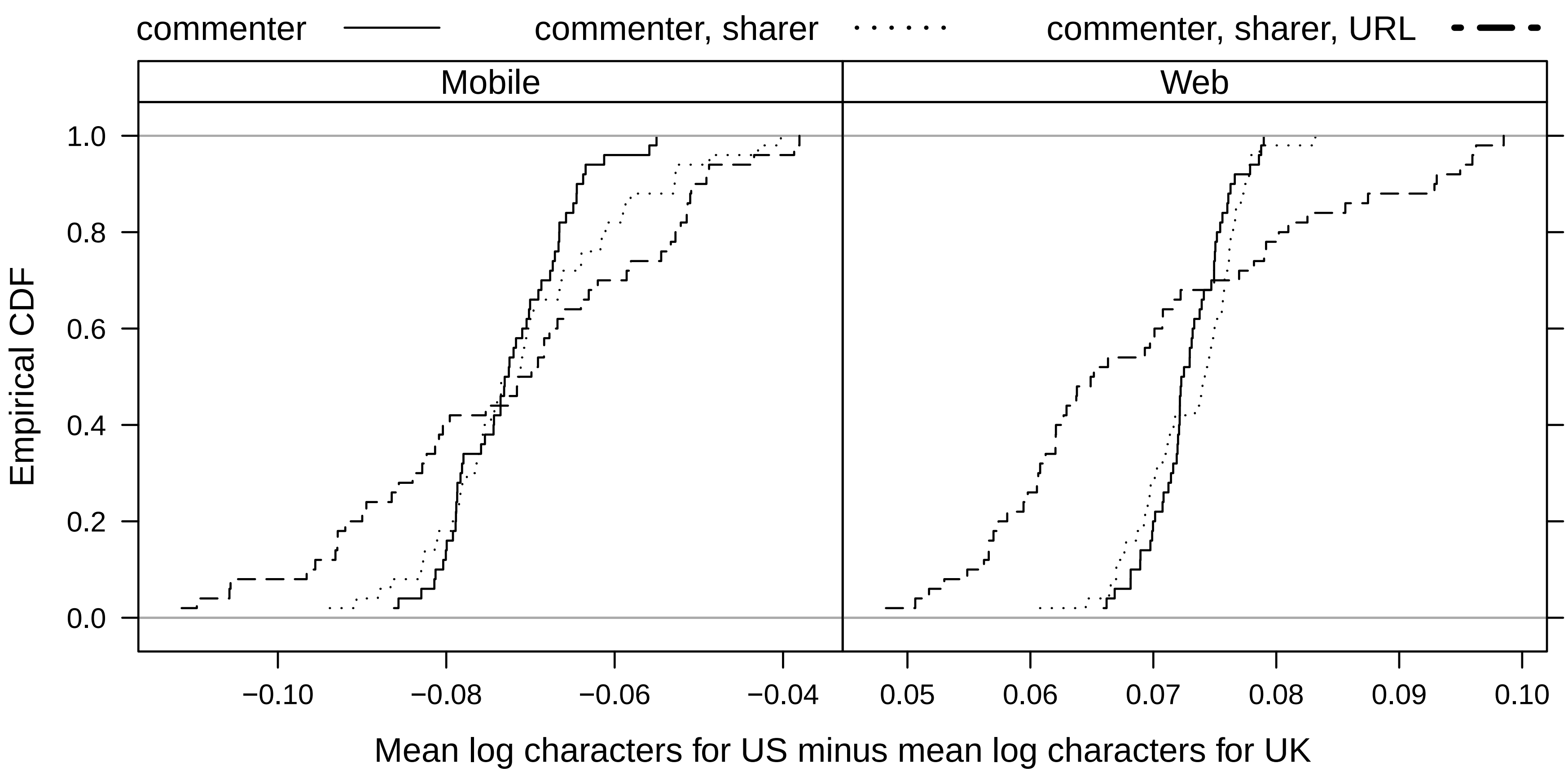
$$\hat{\mu}_{cm} = \frac{\sum_i Z_i X_i 1_{\text{country}=c} 1_{\text{mode}=m}}{\sum_i Z_i 1_{\text{country}=c} 1_{\text{mode}=m}}$$

We see small differences

| | US | UK |
|---|---|---|
| web | 3.62 | 3.55 |
| mobile | 3.50 | 3.57 |

but they're larger than sample fluctuations

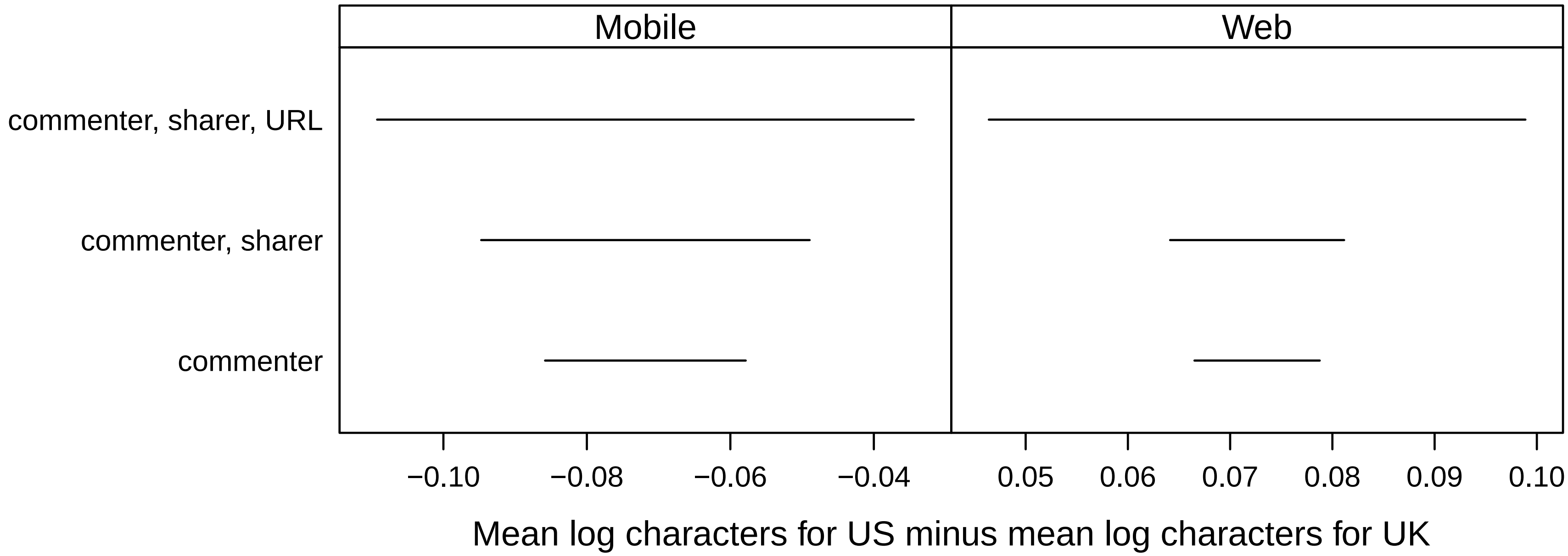
# Loquacity ECDFs




ECDF over $50$ bootstraps of $\hat{\mu}_{\text{US}m} - \hat{\mu}_{\text{UK}m}$

Reweighting one, two, or three ways

# Loquacity confidence intervals




Central $95\%$ confidence intervals from $50$ bootstraps of $\hat{\mu}_{\mathrm{US}m} - \hat{\mu}_{\mathrm{UK}m}$

Reweighting one, two, or three ways

# Heteroscedastic random effects

Every $u \subseteq \{1, 2, \ldots, r\}$ and every $\boldsymbol{i}_u \in \mathbb{N}^{|u|}$ has it's own variance

$$\sigma^2_{\boldsymbol{i},u} \equiv \sigma^2_{\boldsymbol{i}_u,u}$$

We cannot estimate them all.

There may be association between $\sigma^2_{\boldsymbol{i},u}$ and $N_{\boldsymbol{i},u}$.

The analysis now has

$$V_{\text{RE}}(\hat{\mu}) = \frac{1}{N}\sum_u \sum_{\boldsymbol{i}} \nu_{\boldsymbol{i},u}\sigma^2_{\boldsymbol{i},u}, \quad \text{and}$$

$$\mathbb{E}_{\text{RE}}(\widetilde{V}_{\text{PW}}(\hat{\mu}^*)) = \frac{1}{N}\sum_u \sum_{\boldsymbol{i}} \gamma_{\boldsymbol{i},u}\sigma^2_{\boldsymbol{i},u}$$

Product weights still give a mildly conservative variance, with relative error $1 + O(\eta + \epsilon)$ assuming uniform bounds:

$$0 < m \leqslant \min_{\boldsymbol{i},u} \sigma^2_{\boldsymbol{i},u} \leqslant \max_{\boldsymbol{i},u} \sigma^2_{\boldsymbol{i},u} \leqslant M < \infty.$$

# Whence such heteroscedasticity?

Fixed factor $F$ and random mean zero loading $L$

$$X_{\boldsymbol{i}} = \mu + \cdots + F_{i_1} L_{i_2} + \cdots + \varepsilon_{\boldsymbol{i},\{1,\dots,r\}}$$

contributes $F_{i_1}^2 V(L_{i_2})$ to $\sigma^2_{\boldsymbol{i},\{i_2\}}$.
We could have both fixed $i_1$ $\times$ random $i_2$ and vice versa

## More generally

For $v \neq \varnothing$ and $u \cap v = \varnothing$

$$\prod_{j \in u} F_{j,i_j} \times \prod_{j \in v} L_{j,i_j}$$

contributes $\prod_{j \in u} F^2_{h,i_j} \prod_{j \in v} V(L_{j,i_j})$ to $\sigma^2_{\boldsymbol{i},v}$ when $L_{j,i_j}$ are independent.

Factors and loadings don't have to be products

e.g. $F = \Phi(i_1, i_2, i_3)$ fixed & $L = \Lambda(i_4, i_5)$ indep mean $0$

$F \times L$ contributes to $\sigma^2_{\boldsymbol{i},\{4,5\}}$

So the model allows for generalized SVD contributions.

# Gaps and potential next steps

1) The resampler does not imitate the generative model

2) Handling informative missing data

3) Inference for marginal means

$$\bar{X}_{\boldsymbol{i},u} = \frac{\sum_{\boldsymbol{i}'} Z_{\boldsymbol{i}'} 1_{i_u = i'_u} X_{\boldsymbol{i}'}}{\sum_{\boldsymbol{i}'} Z_{\boldsymbol{i}'} 1_{i_u = i'_u}}$$

4) Defining, estimating, and inferring variance components

5) Inference for estimated factor models

6) What about $B = 1$, $B < 1$?

# Thanks

- Dean Eckles for co-authoring
- Netflix and Facebook for data
- Michael Mahoney and other organizers
- NSF DMS-0906056 for support

# The unistrap

Definition $\quad \widetilde{V}_{\mathrm{PW}}(\hat{\mu}^*) \equiv \frac{1}{N^2}\mathbb{E}_{\mathrm{PW}}\big((T^* - \hat{\mu}N^*)^2\big)$

Estimate $\quad \widehat{\widetilde{V}_{\mathrm{PW}}}(\hat{\mu}^*) = \frac{1}{N^2}\frac{1}{B}\sum_{b=1}^{B}(T^{*b} - \hat{\mu}N^{*b})^2$

The $b$'th independent bootstrap produces $(T^{*b}, N^{*b})$ for $b = 1, \dots, B$

Because we're using the ratio estimation formula the estimate exists for $B = 1$.

(and maybe for fractional sampling $B < 1$)

# Modelling $Z_i$

- We do not model the missingness
- Analysis is conditional on $Z_i$
- Make no use/estimate of $X_i$ for $Z_i = 0$

Can/should we do that?

- Missingness is very important
- Less so if you're predicting ratings that were actually made
- Modelling $X_i$ for $Z_i = 0$ requires untestable assumptions (from outside the data)
- Later: use preferred imputation. Resample the result. MC based variance with expert's view of bias.

# Repeated measures

Formally, the model has no duplicate indices

In practice we may get multiple observations at any $\boldsymbol{i}$

We are studying sums for each $\boldsymbol{i}$. This is heteroscedastic (for unequal sample sizes).

## Alternative

We can adjoin an $r+1^{\text{st}}$ index

This index describes a random effect nested within the first $r$ effects

Best to have extra index be a unique data point identifier to avoid large $\epsilon$

We could have $s$ crossed random effects nested within each level of the first $r$ effects

It fits into the model with

$$r' = r + s \quad \text{and} \quad \sigma_u^2 = 0 \quad \text{whenever}$$

$$u \cap \{r+1, \dots, r+s\} \neq \varnothing \quad \text{and} \quad u \cap \{1, 2, \dots, r+s\} \neq \{1, 2, \dots, r+s\}$$

# Exact formula depends on

| Notation | Definition | Meaning |
|---|---|---|
| $N_{\boldsymbol{i},u}$ | $\sum_{\boldsymbol{i}'} Z_{\boldsymbol{i}'} 1_{\boldsymbol{i}_u = \boldsymbol{i}'_u}$ | Match $\boldsymbol{i}$ in $u$ |
| $\nu_u$ | $N^{-1} \sum_{\boldsymbol{i}} Z_{\boldsymbol{i}} N_{\boldsymbol{i},u}$ | Avg # matches on $u$ |
| $M_{\boldsymbol{i}\boldsymbol{i}'}$ | $\{j \mid i_j = i'_j\}$ | Match set for $\boldsymbol{i}$ & $\boldsymbol{i}'$ |
| $N_{\boldsymbol{i},k}$ | $\sum_{\boldsymbol{i}'} Z_{\boldsymbol{i}'} 1_{\lvert M_{\boldsymbol{i}\boldsymbol{i}'}\rvert = k}$ | Match $\boldsymbol{i}$ in **exactly** $k$ places |
| $\rho_k$ | $N^{-1} \sum_{\boldsymbol{i}} Z_{\boldsymbol{i}} N_{\boldsymbol{i},k}$ | Avg # $k$-matches |
| $\nu_{k,u}$ | $N^{-2} \sum_{\boldsymbol{i}} \sum_{\boldsymbol{i}'} Z_{\boldsymbol{i}} Z_{\boldsymbol{i}'} 1_{\lvert M_{\boldsymbol{i}\boldsymbol{i}'}\rvert = k} 1_{\boldsymbol{i}_u = \boldsymbol{i}'_u}$ | Match $k$ places including $u$ |
| $\widetilde{\nu}_{k,u}$ | $N^{-3} \sum_{\boldsymbol{i}} \sum_{\boldsymbol{i}'} \sum_{\boldsymbol{i}''} Z_{\boldsymbol{i}} Z_{\boldsymbol{i}'} Z_{\boldsymbol{i}''} 1_{\lvert M_{\boldsymbol{i}\boldsymbol{i}'}\rvert = k} 1_{\boldsymbol{i}_u = \boldsymbol{i}''_u}$ | Hmmm |
| " | $N^{-1} \sum_{\boldsymbol{i}} N_{\boldsymbol{i},u} N_{\boldsymbol{i},k}$ | |

Exact result $\gamma_u = \sum_{k=0}^{r} (1+\tau^2)^k (\nu_{k,u} - 2\widetilde{\nu}_{k,u} + \rho_k \nu_u)$ non-asymptotic

$$\mathbb{E}_{\mathrm{RE}}(\widetilde{V}_{\mathrm{PW}}(\hat{\mu}^*)) = \frac{1}{N} \sum_{u \neq \varnothing} \gamma_u \sigma_u^2$$

# Some history

Boot-II was called Boot-p,i by Brennan, Harris Hanson (1987)

p,i stands for person, item

They wanted to bootstrap variance component estimates in educational testing (students $\times$ questions).

McCullagh (2000) showed it was impossible

McCullagh (2000) has two different Boot-II algorithms, one for nested data

See also Wiley (2001).