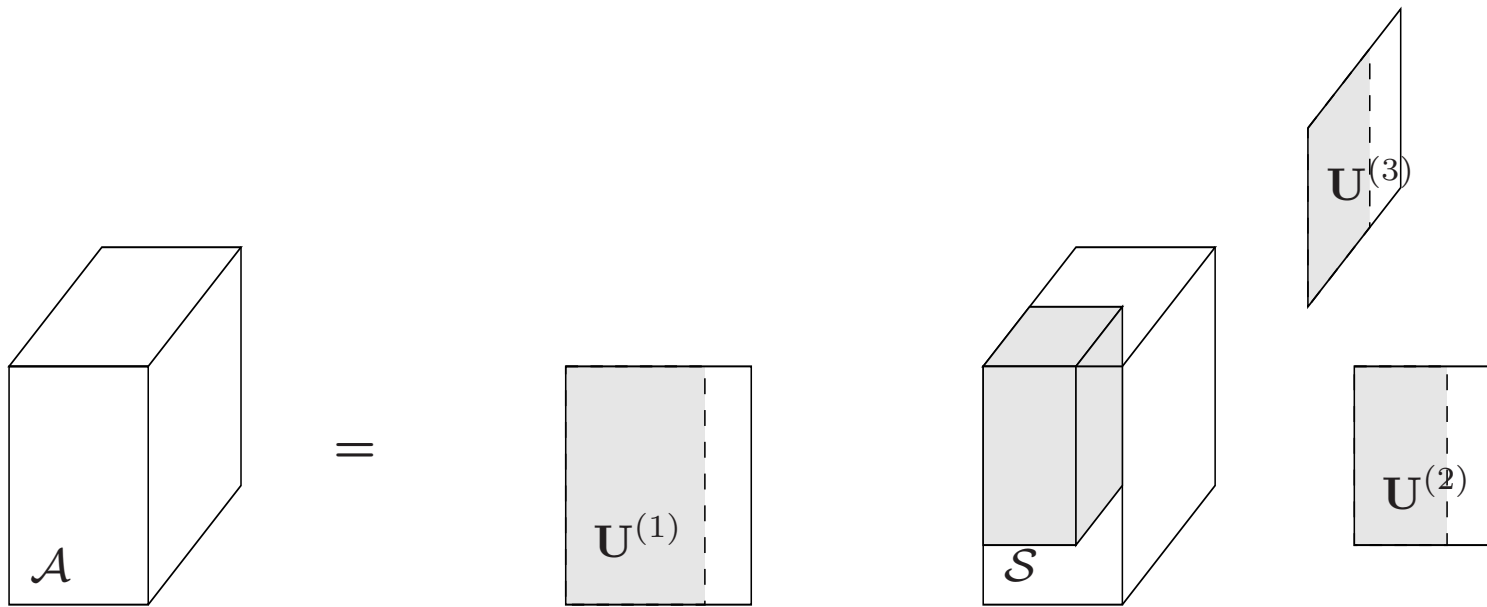


The Decomposition of a Tensor in a Sum of Rank- (R_1, R_2, R_3) Terms

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Tucker's decomposition and best rank- (R_1, R_2, R_3) approximation

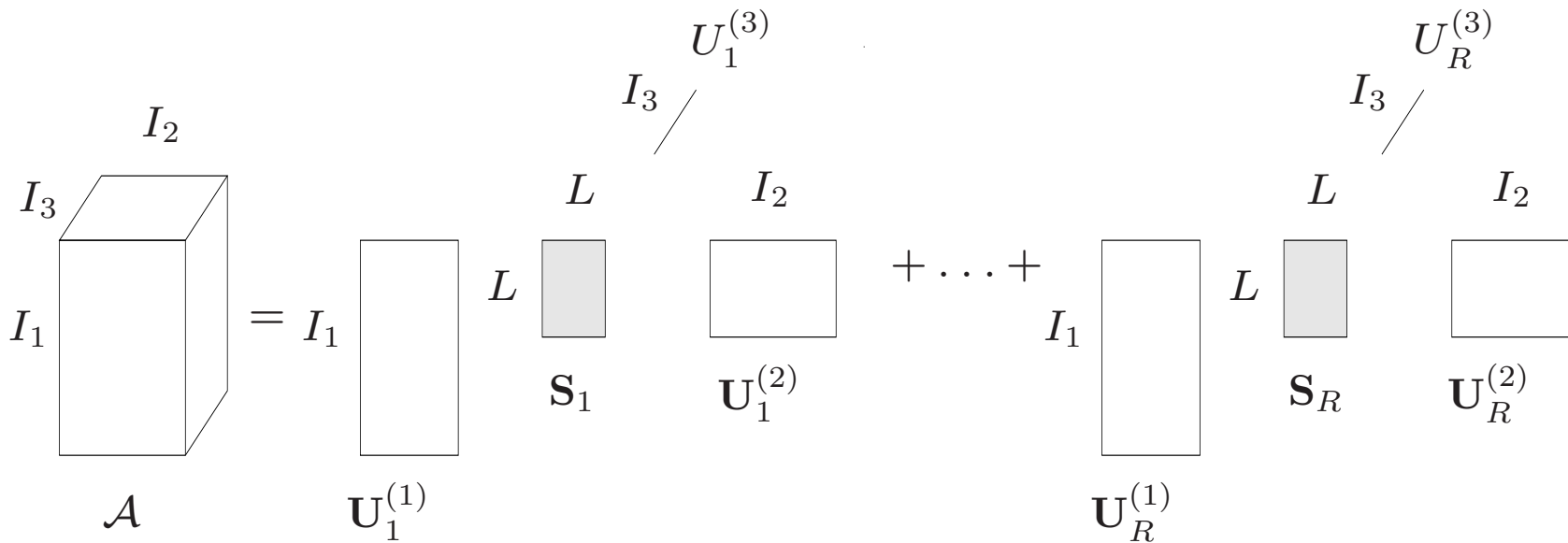


PARAFAC

The diagram illustrates the PARAFAC decomposition of a 3D tensor A . On the left, a 3D box labeled A is shown. To its right is an equals sign, followed by a sum of rank-1 tensors. Each rank-1 tensor is represented by a vertical line on the left, a horizontal line on the bottom, and a diagonal line on the right. The vertical line is labeled $U_1^{(1)}$, $U_2^{(1)}$, or $U_R^{(1)}$ at the bottom. The horizontal line is labeled $U_1^{(2)}$, $U_2^{(2)}$, or $U_R^{(2)}$ on the right. The diagonal line is labeled $U_1^{(3)}$, $U_2^{(3)}$, or $U_R^{(3)}$ at the top. The terms are separated by plus signs, with an ellipsis between the second and third terms.

$$A = U_1^{(1)} U_1^{(2)} U_1^{(3)} + U_2^{(1)} U_2^{(2)} U_2^{(3)} + \dots + U_R^{(1)} U_R^{(2)} U_R^{(3)}$$

Decomposition in rank- $(L, L, 1)$ terms

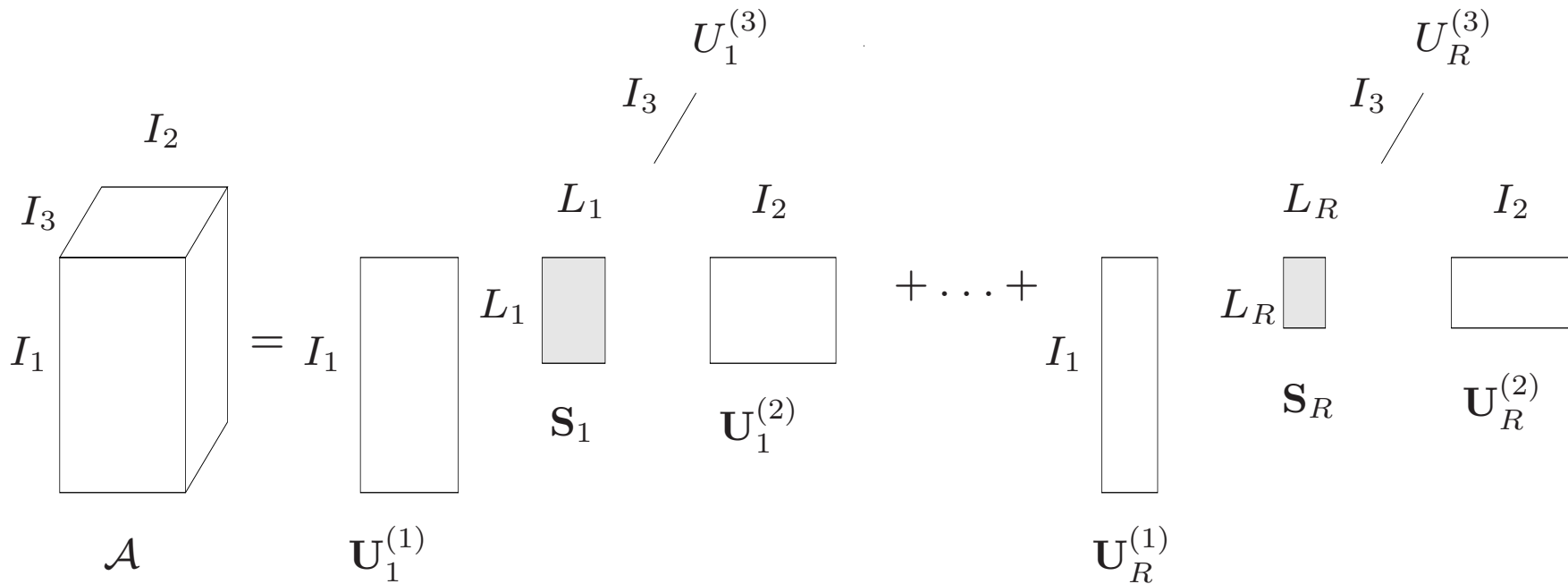


Uniqueness

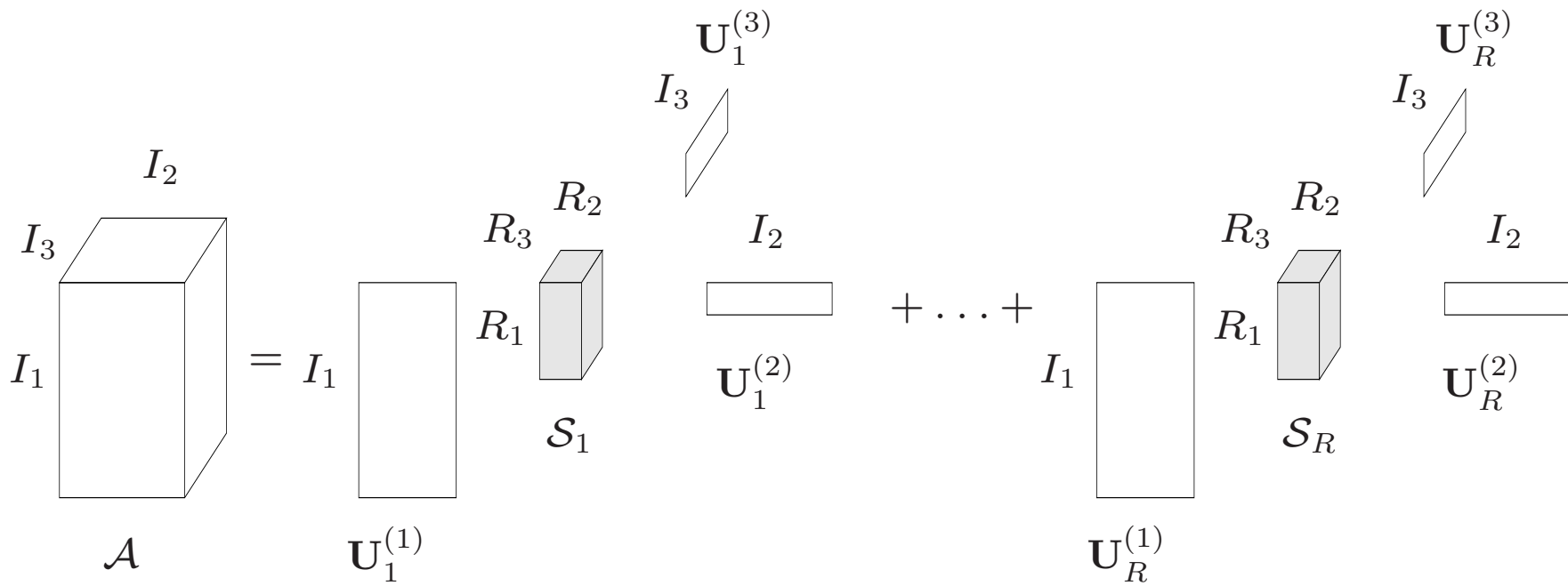
$$\min\left(\left\lfloor \frac{I_1}{L} \right\rfloor, R\right) + \min\left(\left\lfloor \frac{I_2}{L} \right\rfloor, R\right) + \min(I_3, R) \geq 2R + 2$$

cf. $\min(I_1, R) + \min(I_2, R) + \min(I_3, R) \geq 2R + 2$ (PARAFAC)

Decomposition in rank- $(L_r, L_r, 1)$ terms



Decomposition in rank- (R_1, R_2, R_3) terms



A decomposition structure

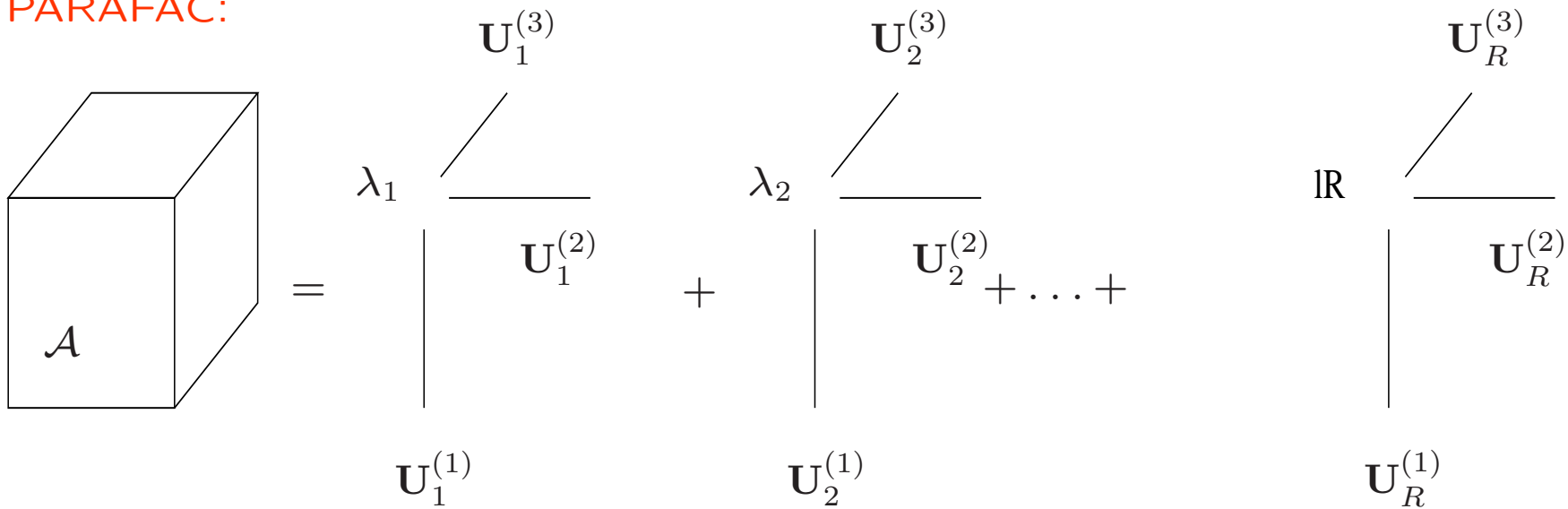
$(I_1 \times I_2 \times I_3)$ tensor \mathcal{A} :

$$\left| \begin{array}{c} (I_1, I_2, I_3) \\ (I_1, I_2, I_3 - 1) \\ (I_1 - 1, I_2 - 2, I_3 - 3) \\ \vdots \\ 1 \end{array} \right| \left| \begin{array}{c} 2(I_1, I_2, \lceil I_3/2 \rceil) \\ 2(I_1, I_2, \lfloor I_3/2 \rfloor) \\ (I_1, I_2, \lfloor I_3/2 \rfloor) + (I_1, \lfloor I_2/2 \rfloor, I_3) \\ \vdots \\ 2(1) \end{array} \right| \cdots \left| \begin{array}{c} R(1) \end{array} \right|$$

- rank
- generalization SVD
- typical rank
- degeneracy
- complex factors cf. [Kaporin, '05]

ALS algorithm

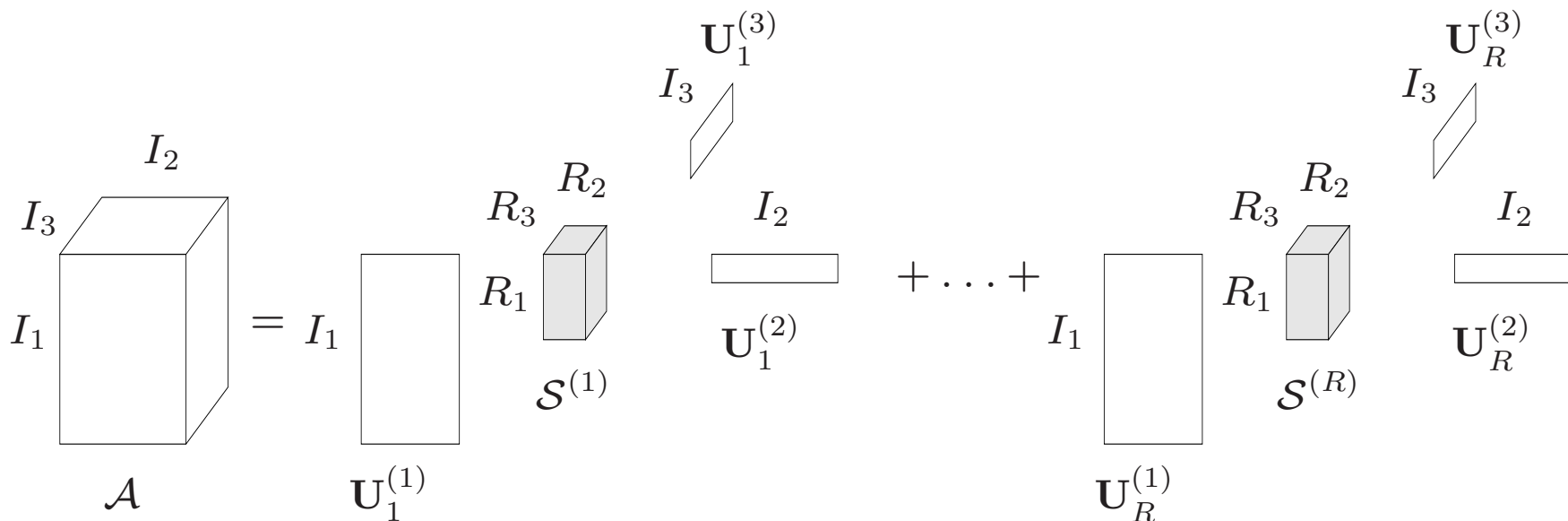
PARAFAC:



$$\mathbf{M}(\{U_r^{(2)}\}, \{U_r^{(3)}\})X = A$$

$$x(1 : I_1) \sim \lambda_1 U_1^{(1)}$$

Decomposition in rank- (R_1, R_2, R_3) terms:



$$\mathbf{M}(\{\mathbf{U}_r^{(2)}\}, \{\mathbf{U}_r^{(3)}\})\mathbf{X} = \mathbf{A}$$

$$x(1 : I_1 R_2 R_3) \sim \mathbf{U}_1^{(1)} \cdot \mathbf{S}^{(1)} \\ (I_1 \times R_1)(R_1 \times R_2 R_3)$$

Perspectives

- Simultaneous matrix decompositions
- Enhanced line search
- Levenberg-Marquardt
- Model order and model structure selection