Near-Optimal Hashing Algorithms for Approximate Near(est) Neighbor Problem

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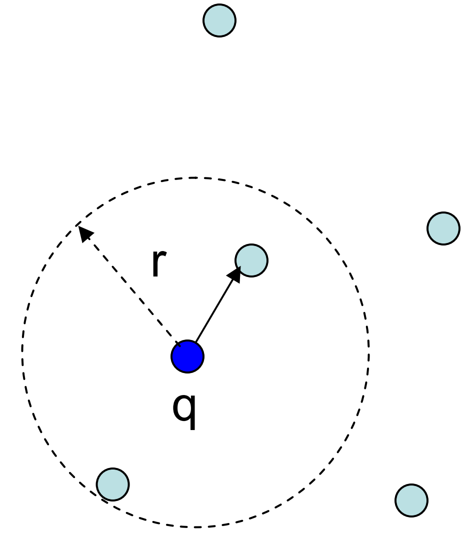
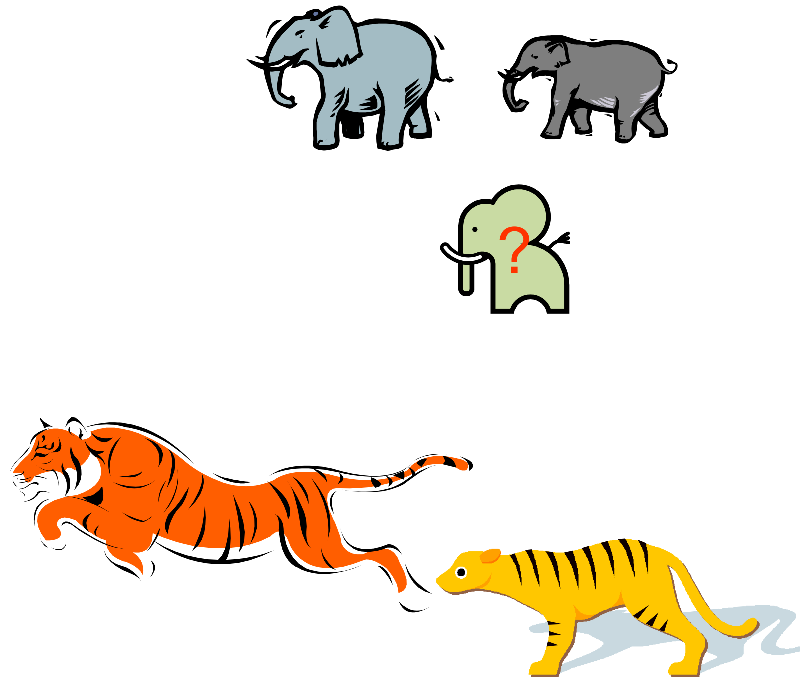
Definition

- Given: a set P of points in R^d
- Nearest Neighbor: for any query q, returns a point $p \in P$ minimizing $||p-q||$
- r-Near Neighbor: for any query q, returns a point $p \in P$ s.t. $||p-q|| \leq r$ (if it exists)



Nearest Neighbor: Motivation

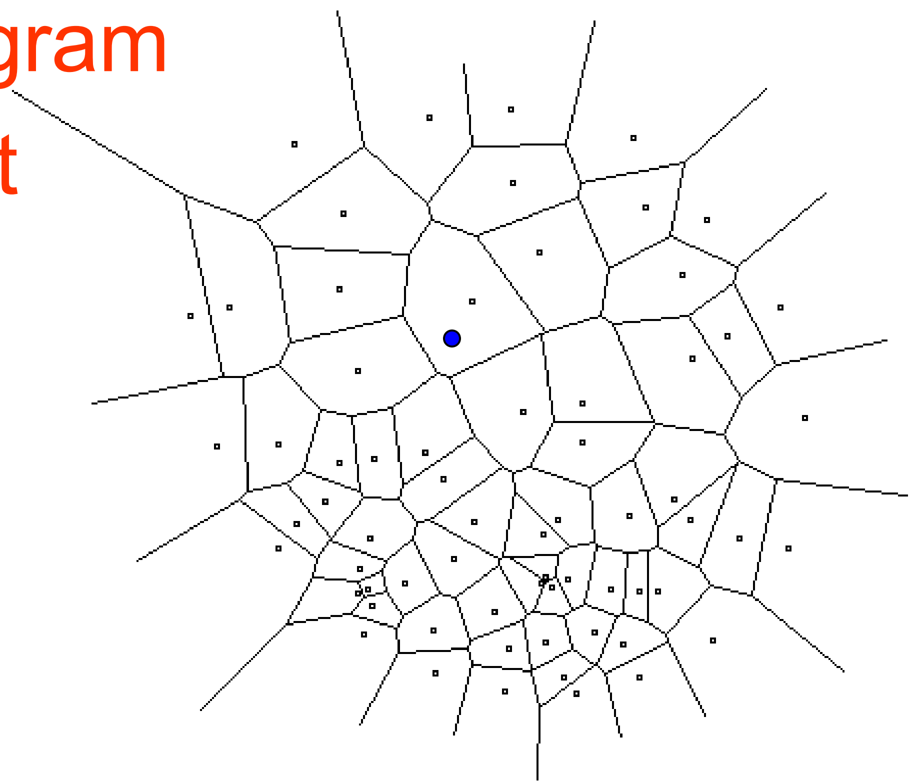
- Learning: nearest neighbor rule
- Database retrieval
- Vector quantization, a.k.a. compression



Brief History of NN

The case of d=2

- Compute Voronoi diagram
- Given q, perform point location
- Performance:
  - Space: O(n)
  - Query time: O(log n)

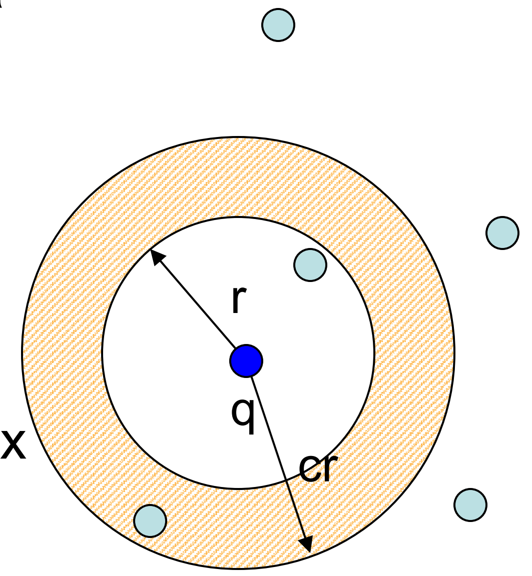


The case of d>2

- Voronoi diagram has size $n^{O(d)}$
- We can also perform a linear scan: $O(dn)$ time
- That is pretty much all what known for exact algorithms with theoretical guarantees
- In practice:
  - kd-trees work "well" in "low-medium" dimensions
  - Near-linear query time for high dimensions

Approximate Near Neighbor

- c-Approximate r-Near Neighbor: build data structure which, for any query q:
  - If there is a point $p \in P$, $||p-q|| \leq r$
  - it returns $p' \in P$, $||p-q|| \leq cr$



- Reductions:
  - c-Approx Nearest Neighbor reduces to c-Approx Near Neighbor

    (log overhead)
  - One can enumerate all approx near neighbors

    → can solve exact near neighbor problem
  - Other apps: c-approximate Minimum Spanning Tree, clustering, etc.

Approximate algorithms

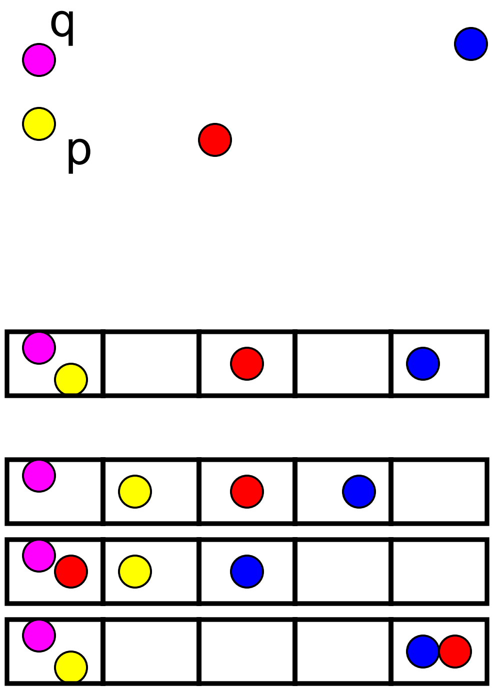
- Space/time exponential in d [Arya-Mount-et al], [Kleinberg'97], [Har-Peled'02], [Arya-Mount-…]
- Space/time polynomial in d [Kushilevitz-Ostrovsky-Rabani'98], [Indyk-Motwani'98], [Indyk'98], [Gionis-Indyk-Motwani'99], [Charikar'02], [Datar-Immorlica-Indyk-Mirrokni'04], [Chakrabarti-Regev'04], [Panigrahy'06], [Ailon-Chazelle'06]…

Space	Time	Comment	Norm	Ref
$dn+n^{4/\varepsilon^2}$	$d * \log n / \varepsilon^2$ or 1	$c=1+\varepsilon$	Hamm, l_2	[KOR'98, IM'98]
$n^{\Omega(1/\varepsilon^2)}$	O(1)			[AIP'06]
$dn+n^{1+\rho(c)}$	$dn^{\rho(c)}$	$\rho(c)=1/c$	Hamm, l_2	[IM'98], [Cha'02]
		$\rho(c)<1/c$	l_2	[DIIM'04]
$dn * \log s$	$dn^{\sigma(c)}$	$\sigma(c)=O(\log c/c)$	Hamm, l_2	[Ind'01]
		$\sigma(c)=O(1/c)$	l_2	[Pan'06]

$dn+n^{1+\rho(c)}$	$dn^{\rho(c)}$	$\rho(c)=1/c^2 + o(1)$	l_2	[AI'06]
$dn * \log s$	$dn^{\sigma(c)}$	$\sigma(c)=O(1/c^2)$	l_2	[AI'06]

Locality-Sensitive Hashing

- Idea: construct hash functions $g: R^d \rightarrow U$ such that for any points p,q:
  - If $||p-q|| \leq r$, then $Pr[g(p)=g(q)]$ is ~~"high"~~ "not-so-small"
  - If $||p-q|| > cr$, then $Pr[g(p)=g(q)]$ is "small"
- Then we can solve the problem by hashing



LSH

[Indyk-Motwani'98,Gionis-Indyk-Motwani'99]

- A family H of functions $h: R^d \rightarrow U$ is called (P_1,P_2,r,cr)-sensitive, if for any p,q:
  - if $||p-q|| < r$ then $Pr[\ h(p)=h(q)\ ] > P_1$
  - if $||p-q|| > cr$ then $Pr[\ h(p)=h(q)\ ] < P_2$
- Example: Hamming distance
  - LSH functions: $h(p)=p_i$, i.e., the i-th bit of p
  - Probabilities: $Pr[\ h(p)=h(q)\ ] = 1-D(p,q)/d$

p=10010010
q=11010110

LSH Algorithm

- We use functions of the form

$$g(p)=<h_1(p),h_2(p),\ldots,h_k(p)>$$

- Preprocessing:
  - Select $g_1 \ldots g_L$
  - For all $p \in P$, hash p to buckets $g_1(p) \ldots g_L(p)$
- Query:
  - Retrieve the points from buckets $g_1(q)$, $g_2(q)$, … , until
    - Either the points from all L buckets have been retrieved, or
    - Total number of points retrieved exceeds $2L$
  - Answer the query based on the retrieved points
  - Total time: $O(dL)$

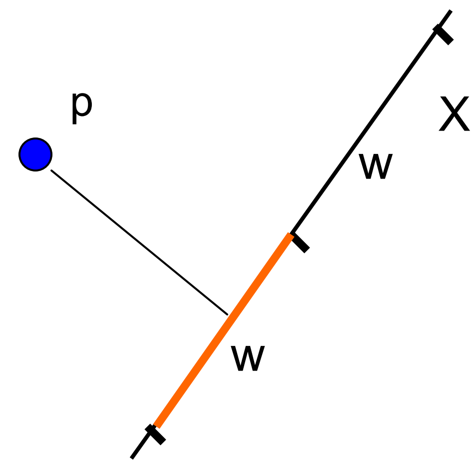
Analysis

- LSH solves c-approximate NN with:
  - Number of hash fun: $L=n^{\rho}$, $\rho=\log(1/P1)/\log(1/P2)$
  - E.g., for the Hamming distance we have $\rho=1/c$
  - Constant success probability per query q
- Questions:
  - Can we extend this beyond Hamming distance ?
    - Yes:
      - embed l_2 into l_1 (random projections)
      - l_1 into Hamming (discretization)
  - Can we reduce the exponent ρ ?

Projection-based LSH

[Datar-Immorlica-Indyk-Mirrokni'04]

- Define $h_{X,b}(p)=\lfloor (p*X+b)/w \rfloor$:
  - $w \approx r$
  - $X=(X_1 \ldots X_d)$, where X_i is chosen from:
    - Gaussian distribution (for l_2 norm)
    - "s-stable" distribution* (for l_s norm)
  - b is a scalar
- Similar to the $l_2 \rightarrow l_1 \rightarrow$ Hamming route

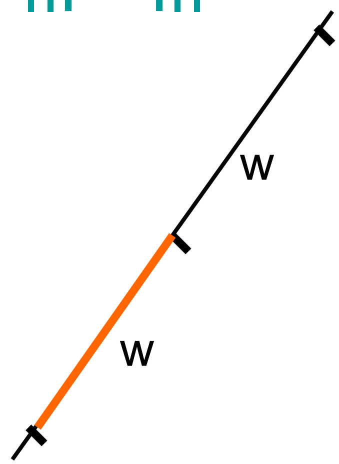


* I.e., $p*X$ has same distribution as $||p||_s \; Z$, where Z is s-stable

Analysis

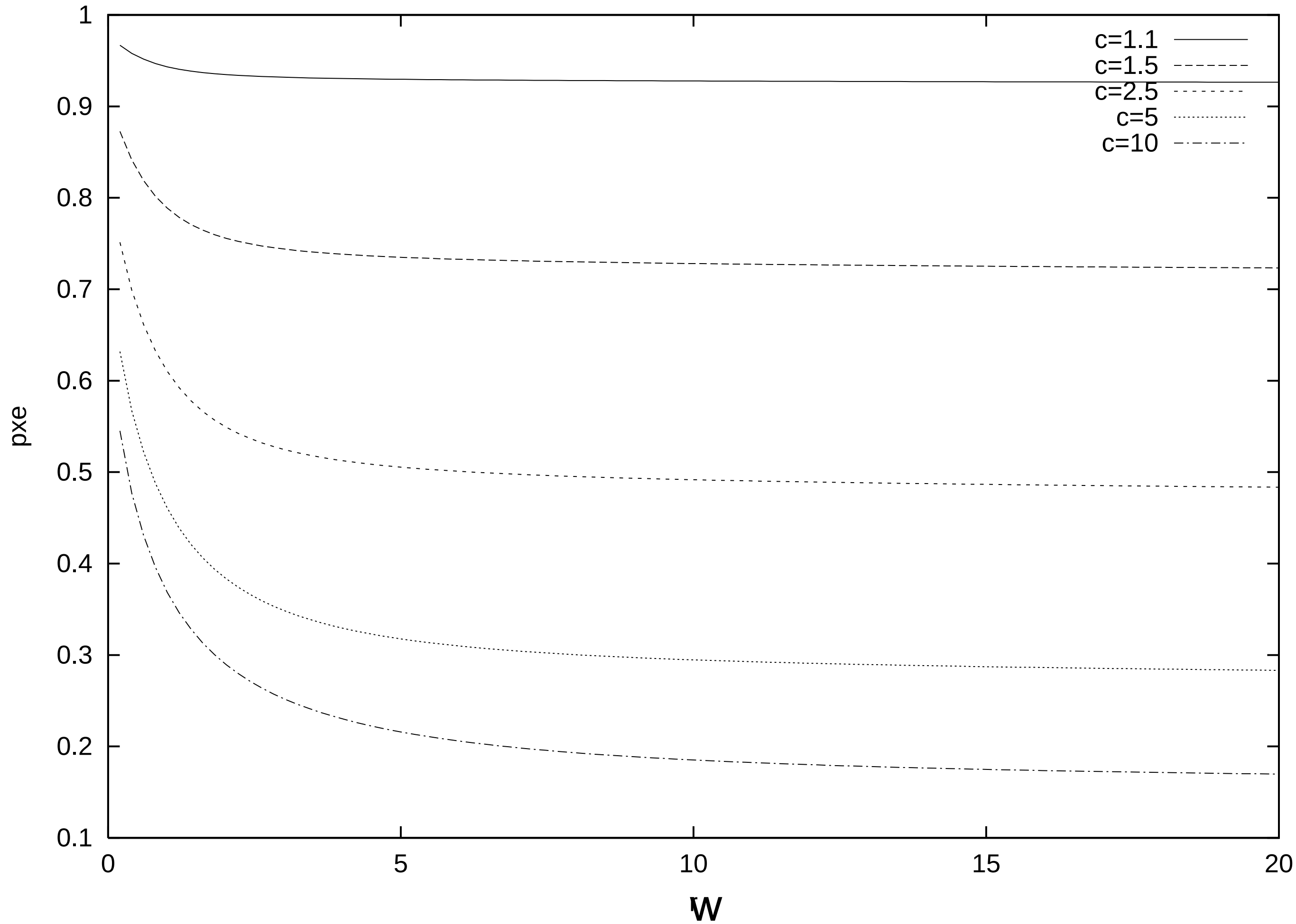
- Need to:
  - Compute $Pr[h(p)=h(q)]$ as a function of $||p-q||$ and w; this defines P_1 and P_2
  - For each c choose w that minimizes

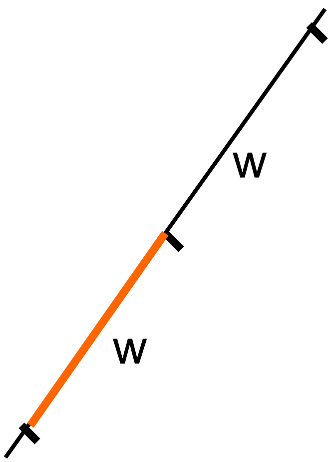
$$\rho=\log_{1/P2}(1/P_1)$$



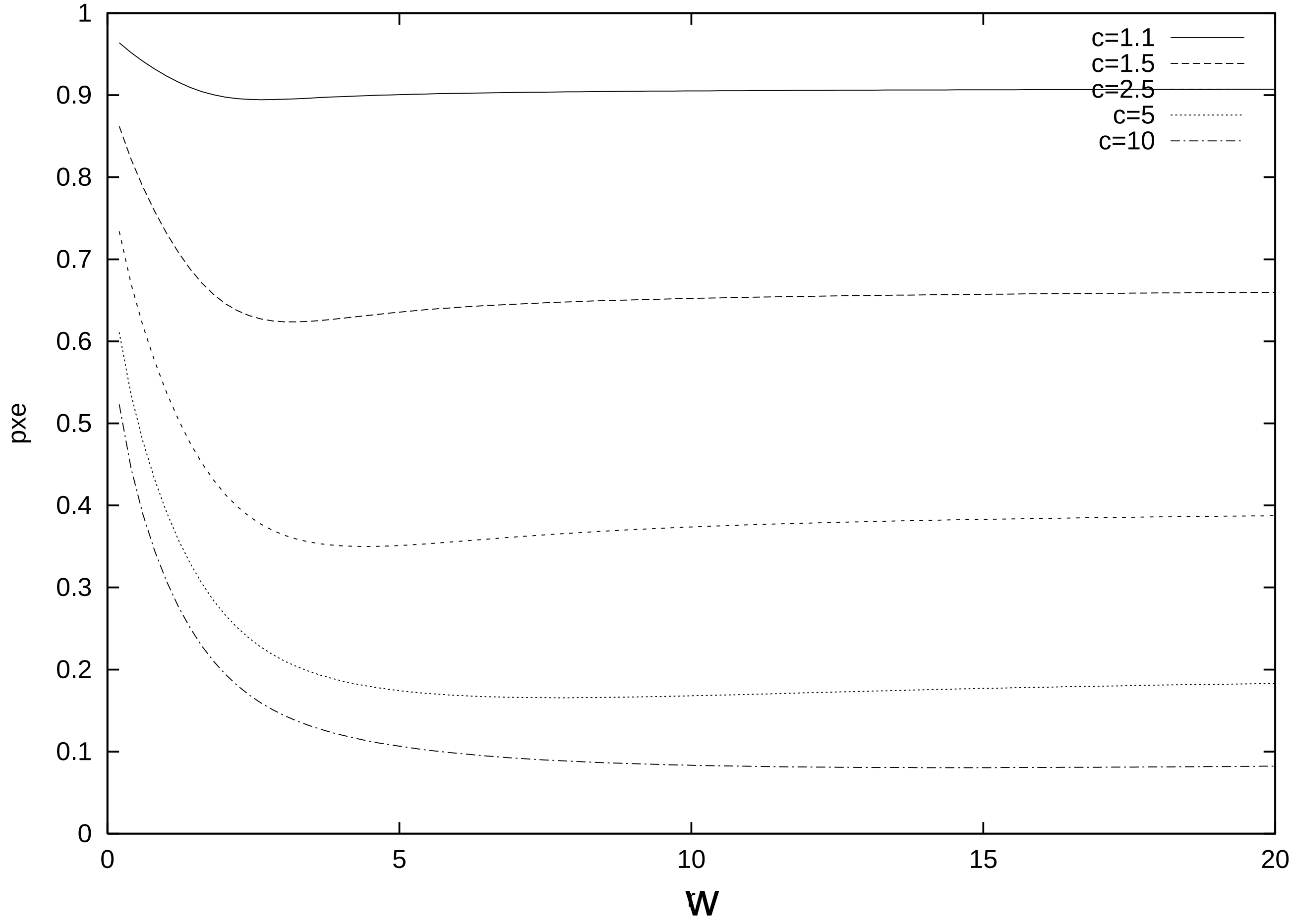
- Method:
  - For l_2: computational
  - For general l_s: analytic

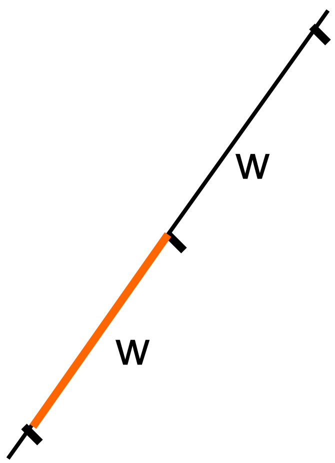
$\rho(w)$ for various c's: l_1



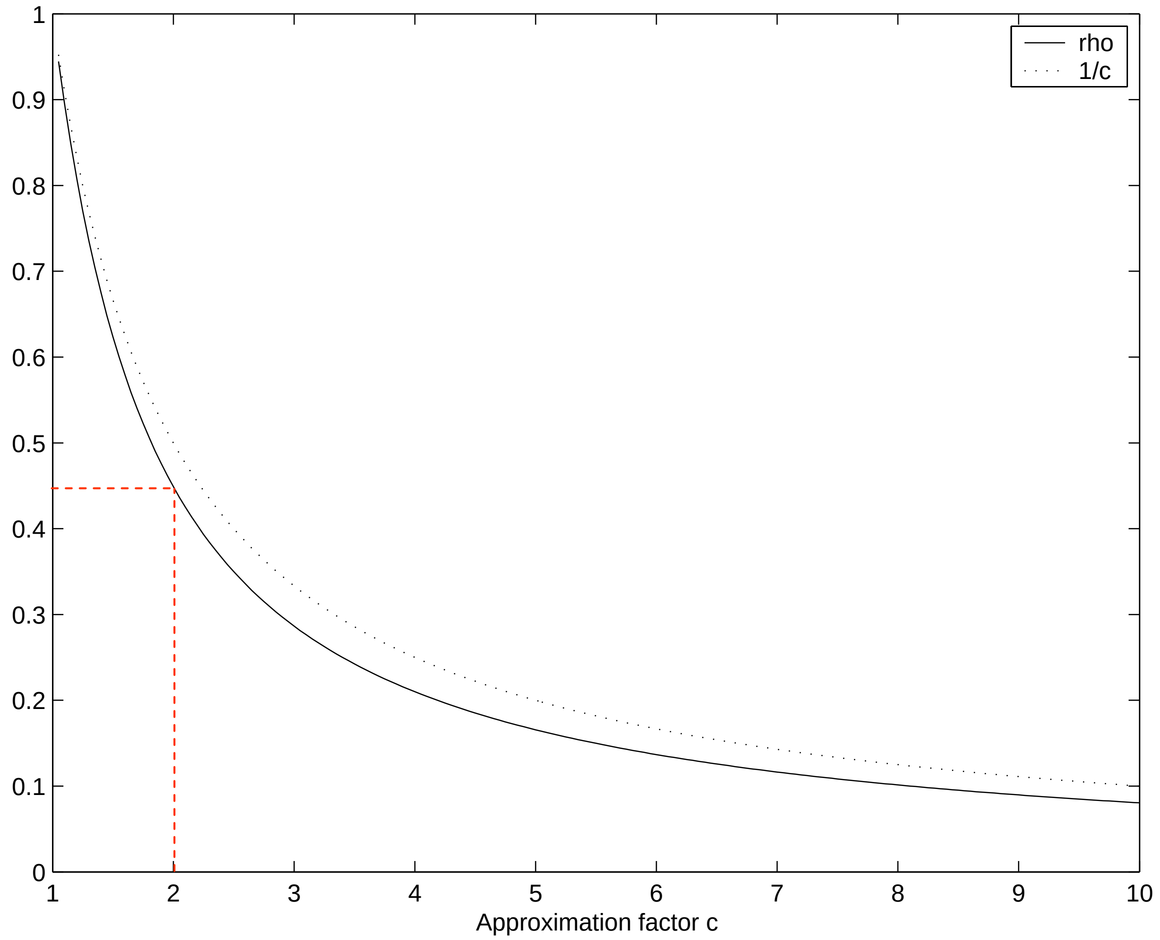


ρ(w) for various c's: l_2





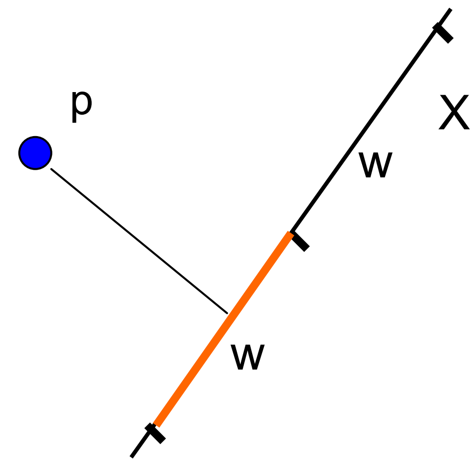
$\rho(c)$ for l_2

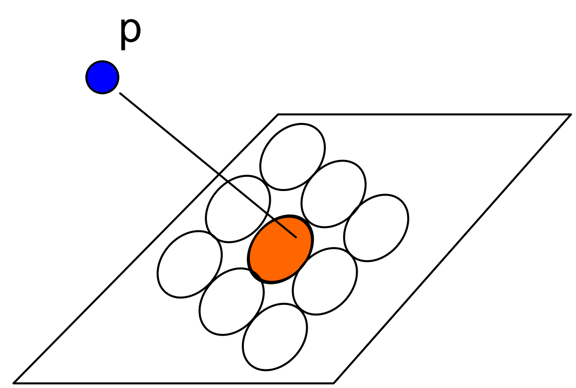

1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
8
9
10
Approximation factor c
rho
1/c

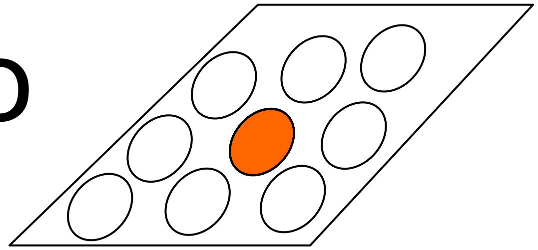
New LSH scheme

[Andoni-Indyk'06]

- Instead of projecting onto R^1, project onto R^t, for constant t
- Intervals → lattice of balls
  - Can hit empty space, so hash until a ball is hit
- Analysis:
  - $\rho = 1/c^2 + O(\log t / t^{1/2})$
  - Time to hash is $t^{O(t)}$
  - Total query time: $dn^{1/c^2+o(1)}$
- [Motwani-Naor-Panigrahy'06]: LSH in l_2 must have $\rho \geq 0.45/c^2$





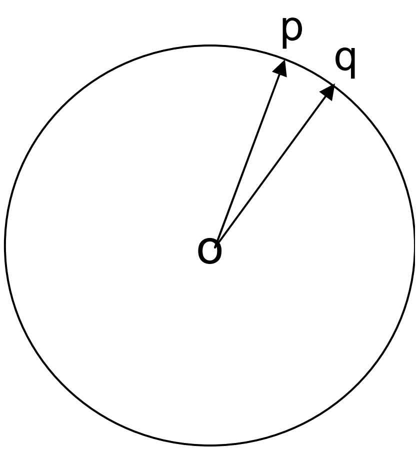
Connections to 

- [Charikar-Chekuri-Goel-Guha-Plotkin'98]
  - Consider partitioning of R^d using balls of radius R
  - Show that $\Pr[\text{Ball}(p) \neq \text{Ball}(q)] \leq ||p-q||/R * d^{1/2}$
    - Linear dependence on the distance – same as Hamming
    - Need to analyze $R \approx ||p-q||$ to achieve non-linear behavior!

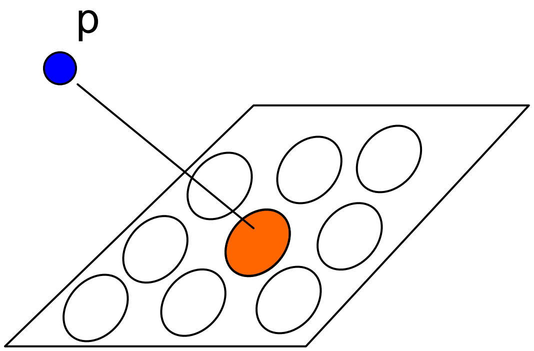
      (as for the projection on the line)
- [Karger-Motwani-Sudan'94]
  - Consider partitioning of the sphere via random vectors u from $N^d(0,1)$:

    $$p \text{ is in } \text{Cap}(u) \text{ if } u*p \geq T$$

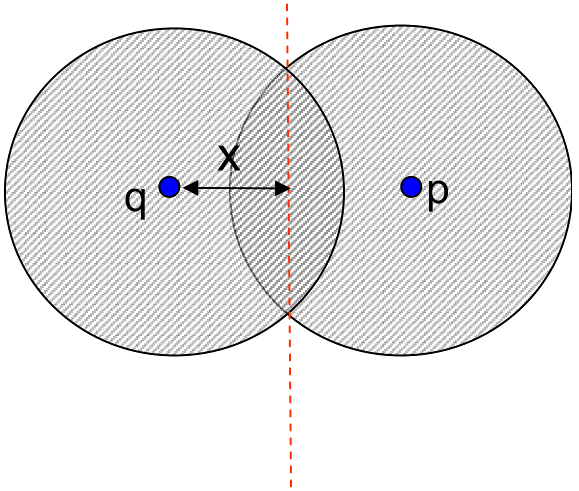
  - Showed $\Pr[\text{Cap}(p) = \text{Cap}(q)] \leq \exp[-(2T/||p+q||)^2/2]$
    - Large relative changes to $||p-q||$ can yield only small relative changes to $||p+q||$



Proof idea

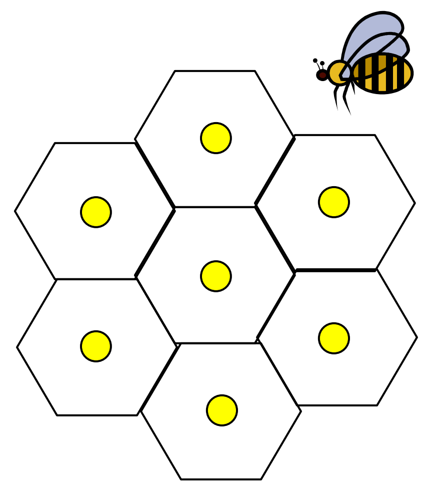


- Claim: $\rho=\log(P1)/\log(P2)\rightarrow 1/c^2$
  - P1=Pr(1), P2=Pr(c)
  - Pr(z)=prob. of collision when distance z
- Proof idea:
  - Assumption: ignore effects of mapping into R^t
  - Pr(z) is proportional to the volume of the cap
  - Fraction of mass in a cap is proportional to the probability that the x-coordinate of a random point u from a ball exceeds x
  - Approximation: the x-coordinate of u has approximately normal distribution
    $\rightarrow$ $Pr(x) \approx \exp(-A\, x^2)$
  - $\rho=\log[\exp(-A1^2)] / \log[\exp(-Ac^2)] = 1/c^2$



New LSH scheme, ctd.

- How does it work in practice ?
- The time $t^{O(t)} d n^{1/c^2 + f(t)}$ is not very practical
  - Need $t \approx 30$ to see some improvement
- Idea: a different decomposition of R^t
  - Replace random balls by Voronoi diagram of a lattice
  - For specific lattices, finding a cell containing a point can be very fast →fast hashing



Leech Lattice LSH

- Use Leech lattice in R^{24} , t=24
  - Largest kissing number in 24D: 196560
  - Conjectured largest packing density in 24D
  - 24 is 42 in reverse…
- Very fast (bounded) decoder: about 519 operations [Amrani-Beery'94]
- Performance of that decoder for c=2:
  - $1/c^2$ 0.25
  - $1/c$ 0.50
  - Leech LSH, any dimension: $\rho \approx 0.36$
  - Leech LSH, 24D (no projection): $\rho \approx 0.26$

Conclusions

- We have seen:
  - Algorithm for c-NN with $dn^{1/c^2+o(1)}$ query time (and reasonable space)
    - Exponent tight up to a constant
  - (More) practical algorithms based on Leech lattice
- We haven't seen:
  - Algorithm for c-NN with $dn^{O(1/c^2)}$ query time and $dn \log n$ space
- Immediate questions:
  - Get rid of the $o(1)$
  - …or came up with a really neat lattice…
  - Tight lower bound
- Non-immediate questions:
  - Other ways of solving proximity problems

Advertisement

- See LSH web page (Google “Locality Sensitive Hashing”):
  - Experimental results (for the ’04 version)
  - Pointers to the code